Thoughts

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1. Monoidal and enriched categories

1.1. Plot synopsis

A long, long time ago[1] I thought up an almost construction to associate an additive category to a semi-additive category and naïvely (and presumptuously) called it an additive completion. At the time I didn't really understand the idea of reflective subcategories or terribly much else[2], but in hindsight the construction concerned the idea of taking the adjunction $G \dashv U : \text{CMon} \to \text{Ab}$ and attempting to elevate it to an adjunction $\underline{G} \dashv \underline{U} : \text{CMon}-\text{Cat} \to \text{Ab}-\text{Cat}$, one between the categories of CMon-enriched and Ab-enriched categories, with the hopes that the fully faithfulness of the underlying adjunction lifted to the new setting.

After some thought, it occurred to me that the capstone of the construction was the fact that left adjoints were cocontinuous. This allowed me to define a composition operation in the new setting, roughly as follows

$$\begin{aligned} i\circ \in &\, \text{CMon}\,(\mathfrak{C}(B,C)\oplus\mathfrak{C}(A,B), UG\mathfrak{C}(A,C)) \\ \cong &\, \text{Ab}\,(G\,(\mathfrak{C}(B,C)\oplus\mathfrak{C}(A,B)), G\mathfrak{C}(A,C)) \\ \cong &\, \text{Ab}\,(G\mathfrak{C}(B,C)\oplus G\mathfrak{C}(A,B), G\mathfrak{C}(A,C)) \ni \circ^+ \end{aligned}$$

In the above, the first isomorphism is due to the adjunction, and the second is due to the cocontinuity of the left adjoint (and the fact that finite products and coproducts coincide for commutative groups and monoids).

Presented as such, the construction offered little hope of generalisation and was unwieldy and largely uninteresting. However, once I recast it in terms of strong monoidal functors commuting with monoidal operations, there appeared a direct avenue to generality. What follows is an attempt at inferring such relationships from a more general context.

1.2. One or 2- categories

We ask the reader to recall the definition of a monoidal category, a (lax) monoidal functor, and a monoidal natural transform.

Prop. 1.2.1. *The collection* MonCat *of monoidal categories, lax monoidal functors and monoidal natural transforms forms a 2-category under the horizontal composition of monoidal functors given by $(F',\phi',\varepsilon')\circ(F,\phi,\varepsilon) = (F'F,(F'\phi)(\phi'(F\times F)),(F'\varepsilon)\varepsilon')$, and the usual horizontal and vertical composition of natural transformations.*

Prop. 1.2.2. *If $\mathfrak{C}$ and $\mathfrak{D}$ are cocartesian monoidal categories and the functor $F : \mathfrak{C} \to \mathfrak{D}$ has a right adjoint, then F is canonically a strong monoidal functor. Moreover, if $\mathfrak{C}$ and $\mathfrak{D}$ are monoidal under biproduct and $F \dashv G : \mathfrak{C} \to \mathfrak{D}$ is an adjunction of functors, then the unit and counit of the adjunction are monoidal natural transforms and the adjunction is monoidal.*

[1] Almost exactly a year

[2] I don't mean to imply that this has changed

We now ask that the reader recall the notions of $\mathfrak{V}$-categories, $\mathfrak{V}$-functors and $\mathfrak{V}$-natural transforms for monoidal category $\mathfrak{V}$.

Prop. 1.2.3. *Let $\mathfrak{V}$ be a monoidal category. The collection $\mathfrak{V}-\textsc{Cat}$ of $\mathfrak{V}$-categories forms a 2-category under the following arrangements, with the evident identity cells:*

1. *Given two composable $\mathfrak{V}$-functors $F : \mathfrak{C} \to \mathfrak{D}$ and $G : \mathfrak{D} \to \mathfrak{E}$, their composite is given by the data $GF : \mathrm{Obj}\,\mathfrak{C} \to \mathrm{Obj}\,\mathfrak{E}$, $(GF)_{A,B} = G_{FA,FB}F_{A,B}$.*

2. *Given composable $\mathfrak{V}$-natural transforms $\beta : F \to G$ and $\gamma : G \to H$ between parallel $\mathfrak{V}$-functors $F, G, H : \mathfrak{C} \to \mathfrak{D}$, their composite is given by $\circ_{FA,GA,HA}(\gamma_A \otimes \beta_A){\lambda^{-1}}_I$.*

3. *Given horizontally composable $\mathfrak{V}$-natural transforms $\beta : F \to F'$ and $\gamma : G \to G'$ between $\mathfrak{V}$-functors $F, F' : \mathfrak{C} \rightrightarrows \mathfrak{D}$ and $G, G' : \mathfrak{D} \rightrightarrows \mathfrak{E}$, their horizontal composite is given by $\circ_{HFA,HF'A,H'F'A}(\gamma_{GA} \otimes H_{FA,GA}\beta_A){\lambda^{-1}}_I$.*

This last proposition was a little painful at times, and I may have cut some corners and employed the proof tactic 'believe me' occasionally, but I trust that the honest reader would never tell anyone. Then, side-stepping size issues,

Def. 1.2.4. Let CatEnrCat denote the subcategory of Cat whose objects are the categories $\mathfrak{V}-\textsc{Cat}$ for $\mathfrak{V}$ ranging among monoidal categories.

Prop. 1.2.5. *The assignment $\underline{\ \ }$: $\textsc{MonCat} \to \textsc{CatEnrCat}$ described below is a 2-functor. On objects $\mathfrak{V}$ it is the assignment $\underline{\mathfrak{V}} = \mathfrak{V}-\textsc{Cat}$, on monoidal functors $F : \mathfrak{V} \to \mathfrak{W}$ it is the assignment of 2-functors $\underline{F} : \mathfrak{V}-\textsc{Cat} \to \mathfrak{W}-\textsc{Cat}$ defined through*

1. *for $\mathfrak{V}$-category $\mathfrak{C}$, $\underline{F}\mathfrak{C}$ is the $\mathfrak{W}$-category with $\mathrm{Obj}\,\underline{F}\mathfrak{C} = \mathrm{Obj}\,\mathfrak{C}$, $\underline{F}\mathfrak{C}(A,B) = F(\mathfrak{C}(A,B))$, composition given by $\circ_{\underline{F}\mathfrak{C}} = F \circ_{\mathfrak{C}} \phi$, and units $j_{\underline{F}\mathfrak{C}} = F j_{\mathfrak{C}} \varepsilon$*

2. *for $\mathfrak{V}$-functor $G : \mathfrak{C} \to \mathfrak{D}$, $\underline{F}G$ is the $\mathfrak{W}$-functor defined by $(\underline{F}G)_{A,B} = F(G_{A,B})$ and the object map $\underline{F}G = G : \mathrm{Obj}\,\mathfrak{C} \to \mathrm{Obj}\,\mathfrak{D}$*

3. *for $\mathfrak{V}$-natural transform η, $\underline{F}\eta$ is the $\mathfrak{W}$-natural transform given by $(\underline{F}\eta)_A = F\eta_A\varepsilon$*

and on monoidal natural transformations $\tau : F \to G$ it is the assignment of natural transformations between 1-functors $\underline{\tau} : \underline{F} \to \underline{G}$ defined through the data $(\underline{\tau}_{\mathfrak{C}})_{A,B} = \tau_{\mathfrak{C}(A,B)}$ and the identity map on objects.

Remark 1.2.6. Note that although the codomain is a 2-category, the change of base functor sends 1-functors to 2-functors. At the time of writing, I'm unsure what 2-natural transforms should be, but I have little doubt that this result can be extended by changing the codomain to 2Cat and showing that $\underline{\tau}$ is a 2-natural transform. ◀

Cor. 1.2.7. *Every adjunction of categories $\mathfrak{C}$ and $\mathfrak{D}$ with biproducts induces an adjunction of $\mathfrak{C}-\textsc{Cat}$ and $\mathfrak{D}-\textsc{Cat}$.*

There is bound to be some relation between monoidal functors F being full, faithful, and or injective on objects and $\underline{F}$ having one or more of these properties. A careful check, I think, would answer the original question.

2. Adjoint squares

There is an exercise in 'Categories Work' which has the reader think about equivalent formulations of what Mac Lane calls an adjoint square – a morphism of adjoint functors. With some thought, this leads to the discovery of a category with a little too much structure. The very next exercise concerns the verification of some isomorphism, but when seen in a specific light, this exercise bears some relation to the previous exercise and its secret.

What follows was originally worked out in the context of categories, functors and adjoints, but there was nothing special about Cat and it was recast it terms of 2-categories.

2.1. Double vision

Def. 2.1.1. An adjoint square from an adjunction $f \dashv g : A \to B$ to $f' \dashv g' : A' \to B'$ is a pair of 1-cells $h : A \to A', k : B \to B'$ and a pair of 2-cells $\sigma : f'h \to kf, \tau : hg \to g'h$, subject to the constraint that at least one of diagrams 2.1a to 2.1d commute.

$$
\begin{tikzcd}
hg \arrow[r, "\tau"] \arrow[d, "\eta' hg"'] & g'k \\
g'f'hg \arrow[r, "g'\sigma g"'] & g'kfg \arrow[u, "g'k\varepsilon"']
\end{tikzcd}
$$

(a)

$$
\begin{tikzcd}
f'h \arrow[r, "\sigma"] \arrow[d, "f'h\eta"'] & kf \\
f'hgf \arrow[r, "f'\tau f"'] & f'g'kf \arrow[u, "\varepsilon' kf"']
\end{tikzcd}
$$

(b)

$$
\begin{tikzcd}
f'hg \arrow[r, "f'\tau"] \arrow[d, "\sigma g"'] & f'g'k \arrow[d, "\varepsilon' k"] \\
kfg \arrow[r, "k\varepsilon"'] & k
\end{tikzcd}
$$

(c)

$$
\begin{tikzcd}
h \arrow[r, "\eta' h"] \arrow[d, "h\eta"'] & g'f'h \arrow[d, "g'\sigma"] \\
hgf \arrow[r, "\tau f"'] & g'kf
\end{tikzcd}
$$

(d)

Diagram 2.1

This is perhaps a curious definition in that we don't currently have any reason to believe that we can tell *which* of the above diagrams commute for a given adjoint square. However, there is a strong statement which resolves this.

Prop. 2.1.2. *Given adjunctions $f \dashv g$ and $f' \dashv g'$, 1-cells $h : A \to A', k : B \to B'$ and 2-cells $\sigma : f'h \to kf, \tau : hg \to g'h$, any one of diagrams 2.1a to 2.1d commute iff they all commute.*

It is difficult to overstate how useful this proposition is in this context – it features in the proof of almost every statement to follow. Furthermore, this proposition has the following curious consequence.

Cor. 2.1.3. *In an adjoint square, σ uniquely determines and is uniquely determined by τ.*

To describe this arrangement, we say that σ and τ are conjugates. At this point, a question presents itself: can the above be phrased in terms of an isomorphism, and perhaps more specifically, are we seeing part of a functorial isomorphism of categories?

Remark 2.1.4. As it happens, in Cat specifically, there is a fifth equivalent condition for 2-cells to form an adjoint square, viz., the commutativity of the following diagram for all objects $C \in \operatorname{Obj}\mathfrak{C}, D \in \operatorname{Obj}\mathfrak{D}$.

$$\begin{tikzcd}
\mathfrak{D}(LC,D) \arrow[r, "\phi_{C,D}"] \arrow[d, "K"'] & \mathfrak{C}(C,RD) \arrow[d, "H"] \\
\mathfrak{D}'(KLC,KD) \arrow[d, "\mathfrak{D}'(\sigma_C,KD)"'] & \mathfrak{C}'(HC,HRD) \arrow[d, "\mathfrak{C}'(HC,\tau_D)"] \\
\mathfrak{D}'(L'HC,KD) \arrow[r, "\phi'_{HC,KD}"'] & \mathfrak{C}'(HC,R'KD)
\end{tikzcd}$$

As we can see, this would be an impossible definition for general 2-categories. ◀

With the conjugate nature of adjoint squares exposed, we are now in a position to examine the rich structure they afford. In order to do so, however, we need to make some minor preparations.

First, we must fix our ambient 2-category and then decide on a suitable means to capture the data of an adjoint square. The above corollary tells us that either of the two conjugates tells the whole picture, and so if we track the adjunction as a single (arbitrarily oriented) arrow, an adjoint square is completely described by a square of 1-cells and a 2-cell spanning its interior. Given that the direction of the adjoint-representative arrow may not necessarily bear meaningful orientation, we choose to omit the arrow indicating the direction of the 2-cell. Of course, we must actually fix the direction of our adjoint arrow and for now we choose to align it with the left adjoint of the pair – a decision whose consequences we shall later examine. All in all, we depict an adjoint square as follows:

$$\begin{tikzcd}
A \arrow[r, "f \dashv g"] \arrow[d, "h"'] \arrow[dr, phantom, "\sigma"] & B \arrow[d, "k"] \\
A' \arrow[r, "f' \dashv g'"'] & B'
\end{tikzcd}$$

Now that we have settled on a visual notation for adjoint squares, we can proceed to ponder their nature. Specifically, consider that given two 'vertically compatible' adjoint squares we may derive a third as follows.

$$\left.\begin{array}{c}
\begin{tikzcd}
A \arrow[r, "f \dashv g"] \arrow[d, "h"'] \arrow[dr, phantom, "\sigma"] & B \arrow[d, "k"] \\
A' \arrow[r, "f' \dashv g'"'] & B'
\end{tikzcd} \\[2em]
\begin{tikzcd}
A' \arrow[r, "f' \dashv g'"] \arrow[d, "h'"'] \arrow[dr, phantom, "\sigma'"] & B' \arrow[d, "k'"] \\
A'' \arrow[r, "f'' \dashv g''"'] & B''
\end{tikzcd}
\end{array}\right\} \to
\begin{tikzcd}
A \arrow[rr, "f \dashv g"] \arrow[d, "h'h"'] & {} \arrow[d, phantom, "(k'\sigma)(\sigma'h)"] & B \arrow[d, "k'k"] \\
A'' \arrow[rr, "f'' \dashv g''"'] & {} & B''
\end{tikzcd}$$

On the surface of things, the claim that the resulting square is an adjoint square is unremarkable – we know that any 2-cell of the correct type gives rise to a conjugate (via diagram 2.1a or diagram 2.1b) with which it forms an adjoint square. Said another way, all we have done above is notice that we may combine two 2-cells of a certain type into a third of the same type. However, under the obvious identity adjoint square (vertical sides identity, interior identity left adjoint) the above operation is unital, and it is always associative. This *is* interesting, we have formed a category.

However, there is another fashion in which we may combine adjoint squares. Specifically, adjoints themselves compose and so we may draw the following picture.

$$\underbrace{
\begin{tikzcd}
A \arrow[r, "f \dashv g"] \arrow[d, "h"'] \arrow[dr, phantom, "\sigma"] & B \arrow[d, "k"] \\
A' \arrow[r, "f' \dashv g'"'] & B'
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\widehat{f} \dashv \widehat{g}"] \arrow[d, "k"'] \arrow[dr, phantom, "\widehat{\sigma}"] & C \arrow[d, "j"] \\
B' \arrow[r, "\widehat{f}' \dashv \widehat{g}'"'] & C'
\end{tikzcd}
}$$

$$\downarrow$$

$$\begin{tikzcd}
A \arrow[rr, "\widehat{f}f \dashv g\widehat{g}"] \arrow[d, "h"'] & {} \arrow[d, phantom, "(\widehat{\sigma}f)(\widehat{f}'\sigma)"] & C \arrow[d, "j"] \\
A' \arrow[rr, "\widehat{f}'f' \dashv g'\widehat{g}'"'] & {} & C'
\end{tikzcd}$$

Again we have found another category structure (here the identities have horizontal edges identity adjoints and 2-cell identity of the vertical edges). But in combination, we have something far more interesting.

Def. 2.1.5. Given a 2-category $\mathfrak{C}$, let $\mathrm{LAd}_v\mathfrak{C}$ denote the category whose objects are adjunctions $f \dashv g$, whose morphisms are adjoint squares (h,σ,k), and whose composition is given by the above-mentioned vertical composition.

Prop. 2.1.6. *By $\partial_0,\partial_1 : \mathrm{LAd}_v\mathfrak{C} \rightrightarrows \mathfrak{C}$ denote the functors defined by $\partial_0(f \dashv g) = \mathrm{dom}\, f$, $\partial_1(f \dashv g) = \mathrm{dom}\, g$, $\partial_0(h,\sigma,k) = h$, $\partial_1(h,\sigma,k) = k$. The assignments $I^h : \mathfrak{C} \to \mathrm{LAd}_v\mathfrak{C}$ and $\bullet : \mathrm{LAd}_v\mathfrak{C} \times_{\mathfrak{C}} \mathrm{LAd}_v\mathfrak{C} \to \mathrm{LAd}_v\mathfrak{C}$ given by $(f \dashv g) \bullet (\widehat{f} \dashv \widehat{g}) = (\widehat{f}f \dashv g\widehat{g})$, $I^hA = (\mathrm{id}_A, \mathrm{id}_A)$ on objects and $(h,\sigma,k) \bullet (k,\widehat{\sigma},j) = (h,(\widehat{\sigma}f)(\widehat{f}\sigma),j)$, $I^hh = (h,\mathrm{id}_h,h)$ on morphisms extend to functors, and endow a double category structure on $\mathrm{LAd}_v\mathfrak{C}$ and $\mathfrak{C}$. We call this double category $\mathrm{LAd}\mathfrak{C}$.*

In fact, more is true. Should we write 2Cat for the 2-category of 2-categories, 2-functors and 2-natural transformations, and VDblcat for the 2-category of double categories, double functors and vertical double natural transformations then

Prop. 2.1.7. *The assignment of objects $\mathrm{LAd} : 2\mathrm{Cat} \to \mathrm{VDblcat}$ given above extends to a 2-functor.*

We are now in a position to return to the question of conjugates. Recall that we made the arbitrary decisions of orienting arrows in the direction of the left adjoint, and choosing σ of the pair to represent the squares. Of course, we can make the complementary decisions, or indeed choose any combination of left- or right- orientation and σ or τ squares for a total of four 2-functors, viz., $\sigma - \mathrm{LAd}$, $\sigma - \mathrm{RAdj}$, $\tau - \mathrm{LAdj}$, and $\tau - \mathrm{RAdj}$.

With these definitions in place, we are one technical lemma away from giving the formal answer to our earlier question.

Lem. 2.1.8. *Given the following adjoint squares,*

$$
\begin{tikzcd}
A \arrow[r, "f \dashv g"] \arrow[d, "h"'] \arrow[dr, phantom, "\sigma"] & B \arrow[d, "k"] \\
A' \arrow[r, "f' \dashv g'"'] & B'
\end{tikzcd}
\quad
\begin{tikzcd}
B \arrow[r, "\widehat{f} \dashv \widehat{g}"] \arrow[d, "k"'] \arrow[dr, phantom, "\widehat{\sigma}"] & C \arrow[d, "j"] \\
B' \arrow[r, "\widehat{f}' \dashv \widehat{g}'"'] & C'
\end{tikzcd}
\quad
\begin{tikzcd}
A' \arrow[r, "f' \dashv g'"] \arrow[d, "h'"'] \arrow[dr, phantom, "\sigma'"] & B' \arrow[d, "k'"] \\
A'' \arrow[r, "f'' \dashv g''"'] & B''
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r, "A \dashv A"] \arrow[d, "h"'] \arrow[dr, phantom, "\mathrm{id}_h"] & A \arrow[d, "h"] \\
A' \arrow[r, "A' \dashv A'"'] & A'
\end{tikzcd}
$$

$(k'\sigma)(\sigma'h)$ and $(\tau'k)(h'\tau)$ are conjugates in the evident vertical composite, $(\widehat{\sigma}f)(\widehat{f}'\sigma)$ and $(g'\widehat{\tau})(\tau\widehat{g})$ are conjugates in the evident horizontal composite, and (h,id_h,h) is conjugate to itself.

We are now finally in a position to answer our question. Writing 'co' for the horizontal opposite of a double category, we see that the assignment of a 2-cell to its conjugate is the 'square-part' of a strict 2-natural isomorphism of 2-functors.

Prop. 2.1.9. *There are strict 2-natural isomorphisms*

$$\sigma - \mathrm{LA{\scriptsize DJ}} \cong \tau - \mathrm{LA{\scriptsize DJ}} \cong (\tau - \mathrm{RA{\scriptsize DJ}})^{\mathrm{co}} \cong (\sigma - \mathrm{RA{\scriptsize DJ}})^{\mathrm{co}}$$

for the 2-functors $2\mathrm{C{\scriptsize AT}} \to \mathrm{VD{\scriptsize BLCAT}}$.

Having answered that, we may return to something which the reader may have thought suspect. We have identified four 2-functors, not three and not five, but four. Incidentally, there are four corners on a square. Do we dare?

$$\begin{tikzcd}
\sigma - \mathrm{LA{\scriptsize D}} \arrow[r, "\lambda"] \arrow[d, "l"'] \arrow[dr, phantom, "\Gamma"] & \sigma - \mathrm{RA{\scriptsize D}}^{\mathrm{co}} \arrow[d, "r"] \\
\tau - \mathrm{RA{\scriptsize D}} \arrow[r, "\rho"'] & \tau - \mathrm{RA{\scriptsize D}}^{\mathrm{co}}
\end{tikzcd}$$

If we could find a way to see these 2-functors as double functors, then there may just be a way to draw the above square with horizontal double natural isomorphisms λ and ρ, vertical double natural isomorphisms l and r, and maybe even a square double natural isomorphism Γ.

Potentially one way to see this would be to consider the construction, given a 2-category $\mathfrak{C}$, of the double category $\mathcal{Q}\mathfrak{C}$ of objects, 1-cells in both axes and 2-cells between the composites of the edges. In this way we could form double categories of 2Cat and VDblcat and it is easy to see that 2-functors will be promoted to double functors in this manner. However, this is probably the least interesting way to realise the above. A more interesting path to take may be that of working over Bicat where the horizontal arrows are lax, the vertical ones are oplax and with some appropriate notion of binatural transform for the squares. This would require some thought as to the nature of adjunctions in general bicategories.

2.2. The curious case of the Cat$(-,-)$

If we adopt the convention of writing $\mathfrak{D}^{\mathrm{coop}}$ for the double category obtained by taking opposites of the arrow and object categories, then we have

Prop. 2.2.1. *The assignment* $[-,-] : (\mathrm{LA{\scriptsize D}}_v\mathrm{C{\scriptsize AT}})^{\mathrm{op}} \times \mathrm{LA{\scriptsize D}}_v\mathrm{C{\scriptsize AT}} \to \mathrm{LA{\scriptsize D}}_v\mathrm{C{\scriptsize AT}}$ *given by*

$$(\mathfrak{C}, L \dashv R, \mathfrak{D}) \times (\mathfrak{C}', L' \dashv R', \mathfrak{D}') \mapsto ([\mathfrak{C},\mathfrak{C}'], [R,L'] \dashv [L,R'], [\mathfrak{D},\mathfrak{D}'])$$

$$(H,K,\sigma,\tau) \times (H',K',\sigma',\tau') \mapsto ([H,H'],[K,K'],[\tau,\sigma'],[\sigma,\tau'])$$

is a functor. Moreover, combined with the functor $[-,-] : \mathrm{C{\scriptsize AT}}^{\mathrm{op}} \times \mathrm{C{\scriptsize AT}} \to \mathrm{C{\scriptsize AT}}$, *these functors extend to a double functor* $[-,-] : (\mathrm{LA{\scriptsize D}}\mathrm{C{\scriptsize AT}})^{\mathrm{coop}} \times \mathrm{LA{\scriptsize D}}\mathrm{C{\scriptsize AT}} \to \mathrm{LA{\scriptsize D}}\mathrm{C{\scriptsize AT}}$.

This is a rather large generalisation of the classical notion that "adjunctions lift to functor categories".

Remark 2.2.2. Notice that the classical 2-category $\textsc{Adj}$ is a subcategory of $\mathrm{LAd}\textsc{Cat}$, specifically, the subcategory obtained by taking only identities as vertical arrows. It is closed under $[-,-]$ and so the functor restricts to $\textsc{Adj}^{\mathrm{op}} \times \textsc{Adj} \to \textsc{Adj}$. ◀

The above proposition can almost be seen as a consequence of the functoriality of LAd, specifically, for a fixed category $\mathfrak{C}$ we have that $[\mathfrak{C},-] : \textsc{Cat} \to \textsc{Cat}$ is a 2-functor and so $\mathrm{LAd}[C,-] : \mathrm{LAd}\textsc{Cat} \to \mathrm{LAd}\textsc{Cat}$ is a double functor. However, something else is happening here. The functor $[-,-]$ has domain $\textsc{Cat}^{\mathrm{op}} \times \textsc{Cat}$, but we managed to induce a functor with domain $(\mathrm{LAd}\textsc{Cat})^{\mathrm{coop}} \times \mathrm{LAd}\textsc{Cat}$ and not, as we may have expected, $\mathrm{LAd}(\textsc{Cat}^{\mathrm{op}} \times \textsc{Cat})$. This raises the question of whether $\mathrm{LAd}\mathfrak{C}^{\mathrm{op}} \cong (\mathrm{LAd}\mathfrak{C})^{\mathrm{coop}}$ and $\mathrm{LAd}(\mathfrak{C} \times \mathfrak{D}) \cong \mathrm{LAd}\mathfrak{C} \times \mathrm{LAd}\mathfrak{D}$ as double categories, and hopefully as composites of (2-) functors.

2.3. Tears & fears

Unfortunately, this is as far as I could take these notions with the knowledge and free time (!) that I had at the time of writing. Arranged in terms of likelihood of truth, the following remain to be explored.

1. I don't know what a 2-adjunction is, but it is likely that $\times$ and $[-,-]$ form one on $\textsc{Cat}$.
2. Assuming the above, I suspect that those two functors are (ordinary) adjoints on $\mathrm{LAd}_v\textsc{Cat}$.
3. Assuming the above, I doubt there is anything special about $\textsc{Cat}$. I don't know what a cartesian closed 2-category is[3], but presumably $\textsc{Cat}$ is one such example and I think it likely that we can safely prove the above in this more general context.
4. I don't know what a double-adjunction should be, but assuming the above, it would seem possible that $[-,-]$ and $\times$ form one on $\mathrm{LAd}\mathfrak{C}$ for any cartesian closed 2-category $\mathfrak{C}$.

Perhaps this is ideas are supported by the fact that LAd is a 2-functor, in the following sense. If a 2-adjunction (specifically cartesian closedness) can be formulated in a unit-counit manner and it makes sense to speak of vertical double adjunctions in the classical unit-counit way then LAd will certainly guarantee the transport of this structure from $2\textsc{Cat}$ to $\mathrm{V}\textsc{Dblcat}$

[3] or for that matter, a monoidal 2-category

3. Universal algebra

There are several exercises in 'Categories Work' which effectively have the reader variously show that the category of X-objects ($X \in$ {monoid, group, &c.}) in Set is complete. In a very strong sense all of these proofs 'do the same thing', though it is not quite clear how to unify them. Moreover, one may wonder how much of this depends on the category Set itself. The question here then is "how does one show that the category of algebraic objects of a fixed type in a fixed category inherit all ambient limits, uniformly in the category and the type?".

A small clue to the answer is given later in text in the form of the definition of the type τ of an algebraic system. However, as presented the notion is not abstract enough (Mac Lane acknowledges this) in that it makes implicit assumptions about underlying sets. Despite these shortcomings, the definition allowed for the interpretation that an algebra of type τ is an *assignment* of functions to the operations in a compatible manner. To our eyes, every good assignment is a actually a functor.

The present form of the work began to take shape when it was realised that the monoid objects in a monoidal category inherit limits from the ambient category *and* $\mathrm{Mon}\,\mathfrak{C} \cong \mathrm{MonCat}(1, \mathfrak{C})$ as categories. What follows is not necessarily an answer to the question, but perhaps is indicative that the approach of thinking of functors as instantiations of categories representing algebraic types is not without merit.

3.1. The type of things to come

This is the difficult part, and if this can be done the rest of the results follow rather easily. Unfortunately I ran out of time before I could give it much thought, but here's the emotional content of the two major constructions that need to happen:[4]

1. An association giving to each precategory $a, c : \Omega \rightrightarrows \mathbb{N}$ a braided strict monoidal category whose monoidal operation is addition of naturals and whose braiding is non-trivial.

2. An association giving to each equivalence relation on the composites, braids and tensors of the generators of the aforementioned braided strict monoidal category a congruence so that a quotient may be taken.

Assuming this possible, we proceed as follows.

Def. 3.1.1. The type of an algebraic system τ is a braided strict monoidal category on objects $\mathbb{N}$ and monoidal operation $+$ with specified arrows Ω and relations thereupon. Given an arrow $\omega : n \to m$ of τ we say that $n = \operatorname{dom}\omega$ is the arity of ω and $m = \operatorname{cod}\omega$ is the co-arity.

[4] I originally thought there may be an avenue to the first construction by way of the free braided strict monoidal category on a category (talk about it, mention 1 and symmetric), and so had a section entitled "I want to braid free". Unfortunately, it doesn't seem to be important, and I'm sad for the loss of that joke.

The braided monoidal structure is crucial for expressing relations between the arrows and for generating our analogue of 'derived operators'. For example, given $f : 2 \to 1$ and $h, g : 1 \rightrightarrows 2$ we think of the composite $f \circ (h + g)$ as expressing the function $f(gx_1, hx_2)$, and commutativity is expressed via the braiding.

Due to the lack of adequate time, this model has an immediate failing in that a braided monoidal structure is insufficient to generate all 'derived operators' in the sense of Mac Lane (and others, no doubt). Specifically, Mac Lane allows for *any* function $n \to m$ to give a resulting substitution pattern of variables, but our braiding alone cannot ever 'increase' or 'decrease' an object, so it seems as though such arbitrary substitutions cannot be emulated. This may be remedied by adjusting the category τ so that $+$ is actually a product, thereby allowing us to generate substitution patterns $n \to m$ for $n > m$ and $n < m$ – those which ignore certain arguments and those which duplicate them – via universal properties.

Next, observe that a braided strong monoidal functor F with domain the strict discrete braided monoidal category $\mathbb{N}$ is determined, up to isomorphism, by its value on 1. That is, for any $n \in \mathbb{N}$, $Fn = F(1 + \ldots + 1) \cong (F1)^{\otimes n}$ with $F0 \cong I$ by definition.

Def. 3.1.2. An algebraic object of type τ, a τ-object, in braided monoidal category $\mathfrak{D}$ is a braided strong monoidal functor $F : \tau \to \mathfrak{D}$. That is, it is an object $A = F1$ of $\mathfrak{D}$, the underlying object, along with arrows ω_A of $\mathfrak{D}$, the structural arrows, given by the composite

$$\omega_A = A^{\otimes n} \to Fn \xrightarrow{F\omega} Fm \to A^{\otimes m}$$

(under some fixed choice of associativity) for $\omega : n \to m$ in τ, where the first and the last isomorphisms arise as the unique composite of the natural isomorphisms $F \otimes F \cong F+$ and the associator of $\mathfrak{D}$. If the functor is faithful, then we term the τ-object faithful.

Due to the naturality of the isomorphism $A^{\otimes n} \cong Fn$, we have that the composite of structural arrows in $\mathfrak{D}$ is equal to the structural arrow of the composite in τ – in symbols $\omega_A {\omega'}_A = (\omega\omega')_A$. Buyer be ware, without faithfulness we cannot meaningfully encode *inequalities* into our type τ because functors are free to disregard them.

Def. 3.1.3. A morphism $h : F \to G$ of τ-objects in a monoidal category $\mathfrak{D}$ is a monoidal natural transform of the corresponding strong monoidal functors, which we term a τ-morphism.

If we suppress isomorphisms for a moment, a morphism of algebras gives a collection of arrows $h^{\otimes n} = {h_1}^{\otimes n} : A^{\otimes n} \to B^{\otimes n}$ which respect the structural arrows in a generalisation of the classical notion $\omega_B(ha_1, \ldots, ha_n) = h\omega_A(a_1, \ldots, a_n)$.

$$\begin{tikzcd}
A^{\otimes n} \arrow[r, "h^{\otimes n}"] \arrow[d, "\omega_A"'] & B^{\otimes n} \arrow[d, "\omega_B"] \\
A^{\otimes m} \arrow[r, "h^{\otimes n}"'] & B^{\otimes m}
\end{tikzcd}$$

Def. 3.1.4. Given a type τ of an algebraic system, we write $\mathrm{Alg}_\tau\, \mathfrak{D}$ for the category $\textsc{MonCat}_{str}(\tau, \mathfrak{D})$ of τ-objects and τ-morphisms in $\mathfrak{D}$.

Remark 3.1.5. Even if the particulars of this construction are untenable, there is at least one immediate and interesting result of thinking of types as categories. While we know that groups are monoids, the corresponding statement would involve a functor *from* the type of monoids *to* the type of groups which would then induce, via precomposition, a functor from groups to monoids.

At its most unsophisticated, this can be seen as a necessary feature of such frameworks by asking the question "to which arrow in the type of monoids shall we send the inversion arrow of groups?". Perhaps more dramatically, should we think of the type of sets as having no arrows whatsoever, it is evident that we could not give a reasonable functor (in our case this is braided monoidal and in general some form of structure respecting functor) from any other non-set type to the type of sets – though we can always do so the other way and consequently every category of τ-objects has a forgetful functor to the ambient category.

Of course, one may see this inherent contravariance from an algebraic perspective. By saying 'groups are monoids' we are saying that all the requisite facets of monoids exist for those of groups, that is, the structure of groups contains within it the structure of monoids and we need only ignore the inversion. Thus, anything defined on all of the arrows of the type of a group may be, in particular, defined on all but the inversion related ones and that is precisely the image of a reasonable functor from the type of monoids. ◀

3.2. Reaching the limit

Our main result is a corollary of two rather general and accessible smaller results about the interaction of functors and limits.

Prop. 3.2.1. *Given a functor $F : \mathfrak{B} \to [\mathfrak{C}, \mathfrak{D}]$, when all the involved limits exist we have $(\lim_{\mathfrak{B}} F)C \cong \lim_{\mathfrak{B}}(E_C F)$ for E_C the evaluation functor.*

The slogan here is that limits in functor categories may be computed pointwise, provided the pointwise limits exist[5]. The above may be reread as a relationship between limits of functors and limits of their underlying their points. In similar vein, one might wonder about the relationship between limits of braided monoidal functors and limits of their underlying ordinary functors.

Prop. 3.2.2. *The forgetful functor $U_{\mathfrak{C},\mathfrak{D}} : \textsc{BrdMonCat}_{lax}(\mathfrak{C}, \mathfrak{D}) \to \textsc{Cat}(\mathfrak{C}, \mathfrak{D})$ creates limits which exist pointwise. Furthermore, under the domain restriction to the subcategory $\textsc{BrdMonCat}_{str}(\mathfrak{C}, \mathfrak{D})$, $U^{str}_{\mathfrak{C},\mathfrak{D}}$ creates limits which exist pointwise and commute with $\otimes_{\mathfrak{D}}$.*

From here it's a straight dash to the finish.

Cor. 3.2.3. *If $\mathfrak{C}$ is a small complete and braided monoidal category then for any type τ the category $\mathrm{Alg}_\tau \mathfrak{C}$ inherits all limits of $\mathfrak{C}$ which commute with $\otimes$.*

In particular, this immediately gives that $\mathrm{Alg}_\tau \textsc{Set}$ is small complete for every τ as $\textsc{Set}$ is small complete and $\times$ is a limit and so commutes with all limits.

[5]The existence of an monic natural transform some of whose components are not monomorphisms is enough to remove any belief that a converse may be true (and further that everything is as it should be).

As much as we would like these results to hold for the subcategory of $\mathrm{Alg}_\tau\mathfrak{C}$ of faithful τ-objects, they do not for our purposes. For these results to be valid we would require the limit over faithful functors to be faithful itself. However, if $\mathfrak{C}$ has a terminal object T then the functor ΔT is terminal in $\mathrm{CAT}(\mathfrak{D},\mathfrak{C})$, and terminal objects are limits over empty diagrams and so over faithful functors in particular. Thus, for our purposes, extra conditions are certainly necessary in order to give this case.

3.3. A more productive approach

Should we wish to rethink matters in terms of τ being a cartesian symmetric strict monoidal category so as to enable an encoding of substitution patterns then, in order to respect this encoding, we would have to move from general strong braided monoidal functors and monoidal natural transforms to finite product preserving functors and natural transforms between them.

The reason we need only consider natural transforms and not monoidal natural transforms is that the universal properties of the product ensure that every such natural transform is, in fact, monoidal.

Should we switch definitions, the key result here would be something like

Prop. 3.3.1. *Braided monoidal functors between cartesian monoidal categories are precisely the finite product preserving functors.*

Should this be true, noting that we automatically have that the monoidal operation commutes with limits because it is one itself, it would follow that $\mathrm{Alg}_\tau\mathfrak{C}$ inherits all limits which exist in $\mathfrak{C}$ a cartesian monoidal category – the desired result.

This is perhaps the most promising avenue, and a full treatment (including addressing the concerns of the two constructions, with adjustments made so that the desired outcome is a cartesian strict monoidal category) would yield a satisfactory answer to the question for all algebraic types definable through (Mac Lane's account of) universal algebras. At the risk of annoying the reader, such an account would surely have appeared here had the author had the time!

3.4. Terminal thoughts

What has been described so far only accounts for what I believe are called 'single-sorted' algebraic systems, that is, where a τ-object is given by a single underlying object. This is a somewhat artificial restriction on our part in that the above results about limits don't make any mention of the nature of the monoidal category τ. Thus, in attempt to extend this to more exotic structures we may be tempted to 'widen' τ – replace it with another discrete-but-for-specified-arrows braided strict monoidal category which is such that there are 'k copies of $\mathbb{N}$' within it, with each 1_j corresponding to an underlying object of our algebraic system. It is not immediately clear to me if this can be made to work. These notions raise the possibility of seeing a general braided monoidal category as some kind of description of an algebraic system.

Such ideas aside, the whole approach of braided strict monoidal categories is not without its limitations. For example, one might be tempted to use this framework to give a proof that the category of internal categories inherits limits from its ambient category. While this is true, it is not provable here as composition is not a total operation (even if we take the likely required arrows-only view to the type of categories)[6]. Moreover, it would seem that we cannot generate all derived operators using the braiding and tensor alone. Finally, we have not been able to satisfactorily give an account of limit inheritance for faithful τ-objects resulting in the dramatic situation wherein the terminal category has a τ-object for every τ, and they're all identical!

[6]This case is frustrating as even if we do work in the pullback-monoidal category of precategories over a given object, not all functors are morphisms of monoids in this sense.