Type Theory and Categories

the unbearable likeness of being

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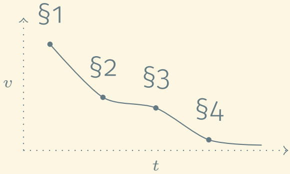
The journey ahead

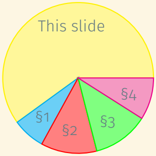
1. Homotopy Type Theory

2. Univalence

3. Category Theory

4. Rezk Completion





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Motivation

- Why should there be a 'the' foundation?
  - Euclid – who cares if l 'is' $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid \dots\}$
  - Arithmetic – who cares if $2 \in 3$
- Set Theory™ does not understand equality of *structures*:
  - groups $G \cong H$
  - "everything that's true of G is true of H"
  - vs $\emptyset \in G$
- In Set Theoretic foundations like ZF:
  - Technically can't *construct* anything
  - Infeasible to actually work *in* the foundations
  - Nothing about *structures* to be gained by doing so

Our foundations should reflect how we do mathematics

The single most exciting thing about this

A technical argument by a trusted author, which is hard to check and looks similar to arguments known to be correct, is hardly ever checked in detail.

A foundation amenable to $\begin{cases} \text{computer verification} \\ \text{proof assistance} \end{cases}$

Proofs *are* programmes, and may be run!

Homotopy Type Theory

You are already a type theorist

Instead of ‘everything is set’, why not have different types which reflect the *qualitatively* different partitions into which we naturally place things?

N	type of natural numbers
Z	type of integers
Mon	type of monoids
0	the ‘empty’ type
1	the ‘singleton’ type
U	the ‘universe’ type $\leftarrow \mathrm{B}$

A typical solution

Basics

- Types have terms, $0 : \mathsf{N}, 1 : \mathsf{N}, \ldots$, and terms belong to a *unique* type.
- Given types A, B, we can form more types: $A \times B$, $A + B$, $A \to B$, ...
- These new types behave as one might expect, if we have $a : A, b : B$ then $(a, b) : A \times B$.
- Types come equipped with certain functions: $\mathsf{succ} : \mathsf{N} \to \mathsf{N}$, $\mathsf{pr_A} : A \times B \to A$, $\mathsf{app_{A,B}} : A \times (A \to B) \to B$.

Note: the statement '$a : A$' is not a proposition, it may not be proven or disproven, it is data.

Is that all? Well it depends...

Definition

We allow ourselves to form the type $A \to \mathsf{U}$. Terms B of this type are *dependent types* or *type families* varying over A.

We also extend the product forming operation:

Definition

Given $A : \mathsf{U}$ and $B : A \to \mathsf{U}$, we define

- the *dependent product*, $\prod(a : A), B(a)$, to be the type comprising terms $f :(a : A) \to B(a)$
- the *dependent sum*, $\sum(a : A), B(a)$, to be the type comprising terms $(a, b : B(a))$

Propositions as types, or, a pronunciation guide

Idea: encode logic statements as types, proofs as terms.

Logic	Types
Proposition on A	$A \to \mathsf{U}$
$P(a)$	$P(a)$
$A \implies B$	$A \to B$
$A \wedge B$	$A \times B$
$A \vee B$	$A + B$
$\forall a[P(a)]$	$\prod(a : A), P(a)$
$\exists a[P(a)]$	$\sum(a : A), P(a)$

"Proof relevance"

How should we express equality?

P.A.T. $\rightsquigarrow$ for $a, b : A$ there is a type $a =_A b$.

Equality is detected by relations $R : A \to (A \to U)$, but these must be *reflexive*.

“All equations are lies…or useless”

Reflexivity: $\mathsf{refl}_a : a =_A a$

Universal property:

$\prod(a : A), R(a, a, \mathsf{refl}_a) \simeq \prod(a, b : A), \prod(p : a =_A b), R(a, b, p)$

Definition

With $a, b, c : A$, $p : a =_A b, q : b =_A c$ and the universal property we can define

- $p^{-1} : b =_A a$
- $p \cdot q : a =_A c$

And we may give terms witnessing

- lunit : $\mathsf{refl}_a \cdot p = p$
- linv : $p^{-1} \cdot p = \mathsf{refl}_a$
- assoc : $(p \cdot q) \cdot r = p \cdot (q \cdot r)$

The analogy

Note: these are *not* ‘strict’ equalities, we can only prove them as propositions in our type theory.

Type Theory	Category Theory	Homotopy Theory
types A, B	∞-groupoids	spaces
functions $f : A \to B$	∞-functors	continuous maps
equality $p : a = b$	equivalence	path
reflexivity $\mathsf{refl}_a : a = a$	identity	constant path
symmetry $p^{-1} : b = a$	inverse	path reversal
transitivity $p \cdot q : a = c$	composition	path concat.

Instances of the analogy

Functions are ∞-functors

We think of types as ∞-groupoids.

A function $f : A \to B$ acts on morphisms:

$$\mathsf{ap}_f(a, b) : (a =_A b) \to (fa =_B fb)$$

Transport

We think of $\mathsf{pr}_1 : (\sum(a : A), B) \to A$ as a fibration.

A path $p : a =_A a'$ in the base space acts on fibres:

$$\mathsf{transport}_{A,B}(a, a', p) : B(a) \to B(a')$$

Univalence

Function extensionality

Given $f, g : A \to B$, what should $f = g$ be?

Definition

Given $f, g : \prod(a : A), B$ define $f \sim g :\equiv \prod(a : A), (fa =_B ga)$ as the type of *homotopies* between f and g.

"all functions are continuous/∞-functors"

Definition

Function extensionality is the *axiom* that there is a term

$$\mathsf{funext} : f \sim g \to f = g.$$

Type extensionality

The analogous question for types

Given $A, B : \mathsf{U}$, what should $A =_{\mathsf{U}} B$ be?

We have a notion of equivalence for types $A \simeq B$, *and* a canonical term idtoeqv $:(A =_U B) \to (A \simeq B)$

Definition

Univalence is the *axiom* that idtoeqv is an equivalence. We name its quasi-inverse

$$\mathsf{ua} :(A \simeq B) \to (A =_{\mathsf{U}} B).$$

Some repercussions

Facts

- Univalence implies function extensionality
- Consistent to assume (Voevodsky's model in sSET)
- Consequently, for any proposition $P : \mathsf{U} \to \mathsf{U}$ and witness $A \simeq B$, one cannot show $P(A)$ but not $P(B)$.

- By the *structure identity principle*, this means that many algebraic things are univalent too.

To get a flavour of the S.I.P., let's pick our favourite toy algebraic structure, ~~affine schemes~~ 'sets' with a binary operation

$$\mathsf{Magma} :\equiv \sum (A : \mathsf{Set}), A \to (A \to A)$$

Audience Participation

$$(f : A \simeq B,\ t : \prod (a, b : A), f(a \star b) =_B (fa) \oplus (fb))$$

$\overset{\text{happly}}{\underset{\text{funext}}{\longleftrightarrow}}$ $$(f : A \simeq B,\ s : (f \star =_{B \to (B \to B)} \oplus (f \times f)))$$

$\overset{\text{idtoeqv}}{\underset{\text{ua}}{\longleftrightarrow}}$ $$(q : A =_{\mathsf{U}} B,\ r : \mathsf{transport}(q, \star) =_{B \to (B \to B)} \oplus)$$

$\overset{\text{generic}}{\underset{\text{lemmas}}{\longleftrightarrow}}$ $$p : (A, \star) =_{\mathsf{Magma}} (B, \oplus)$$

Category Theory

Definition

A *precategory* is a type C_0 equipped with the following

1. $C : \prod(a, b : C_0), \mathsf{Set}_{\mathsf{U}}$
2. $\circ : \prod(a, b, c : C_0), C(b, c) \to C(a, b) \to C(a, c)$
3. $\mathsf{id} : \prod(a : C_0), C(a, a)$
4. $\circ\text{-}\mathsf{id} : \prod(a, b : C_0), \prod(f : C(a, b)), f \circ \mathsf{id}_a = f$
5. ...

Definition

Given precategories A and B, a *functor* $F : A \to B$ comprises the data

1. $F_0 : A_0 \to B_0$
2. $F_{-,-} : \prod(a, b : A_0), A(a, b) \to B(F_0 a, F_0 b)$
3. $F\text{-}\mathsf{id} : \prod(a : A_0), F_{a,a}\mathsf{id}_a = \mathsf{id}_{F_0 a}$
4. ...

Implications of proof relevance

- Precategories are fine, and we can redevelop everything we might care about:
    - Functors are still determined by objects and arrows
    - Natural transformations are still determined by components
    - Adjunctions, Equivalences, Yoneda
    - ...
- **BUT** the following are only (coherent) propositional equalities
    - $G(FH) = (GF)H$ – axioms are part of data and must agree
    - $F\mathsf{id} = F$
    - $(\beta F')(G\alpha) = (G'\alpha)(\beta F) \rightsquigarrow \cancel{\beta \star \alpha}$.

Comparing precategories

1. Equality, $A =_{\mathsf{Precat}} B$
2. Isomorphism, $A \cong_{\mathsf{Precat}} B$
    - $F : A \to B$ is fully faithful
    - $F_0 : A_0 \simeq B_0$
3. Equivalence, $A \simeq_{\mathsf{Precat}} B$
    - $F : A \to B$ is fully faithful
    - *Split* essentially surjective: $\prod(b : B_0), \sum(a : A_0), F_0 a \cong b$
4. Weak equivalence, $A \sim_{\mathsf{Precat}} B$
    - $F : A \to B$ is fully faithful
    - *Merely* essentially surjective: $\prod(b : B_0), ||\sum(a : A_0), F_0 a \cong b||$

We always have 1. $\to$ 2. $\to$ 3. $\to$ 4.

Comparing comparisons

Theorem [Another S.I.P.]

Univalence gives a witness of type $(A =_{\mathsf{Precat}} B) \simeq (A \cong_{\mathsf{Precat}} B)$

Equivalence of precats is strictly weaker than isomorphism

Example

Let A be inhabited and non-contractible, define $(A_{\mathsf{ind}})_0 :\equiv A$ and $A_{\mathsf{ind}}(a, b) :\equiv \mathsf{1}$. The unique functor $A_{\mathsf{ind}} \to \mathsf{1}$ is an equivalence but not an isomorphism.

Categorical approach

The last example runs counter to our intuitions:

A categorical divison

Good concepts are precisely those which are invariant under equivalence. everything else is evil

Lowest dimensional case: nothing should be able to discern isomorphic objects.

$$(a \cong_C b) \simeq (a =_{C_0} b)$$

Definition

A *category* C is a precategory wherein the canonical function $\mathsf{idtoiso}_C : \prod(a, b : C_0), (a =_{C_0} b) \to (a \cong_C b)$ is an equivalence.

"Isomorphic objects are equal"

Examples

- $\mathbf{0}$ and $\mathbf{1}$ are categories
- Univalence implies that $\mathsf{Set}_{\mathsf{U}}$ is a category

Facts

For precategory A and category C,

1. $[A, C]$ is a category
2. $\mathsf{Alg}_C(T)$ is a category for any monad T (S.I.P. !)
3. For a functor $F : C \to A$, $\mathsf{isLeftAdj}(F)$ is a mere prop.
    - Classically, $(G, \eta, \epsilon), (G', \eta', \epsilon') : \mathsf{isLeftAdj}(F)$ gives $G \cong G'$
    - By 1. and some work, $(G, \eta, \epsilon) =_{\mathsf{isLeftAdj}(F)} (G', \eta', \epsilon')$

Categories are not evil

Amazing facts

1. B $\mathsf{isWeakEquiv}(F) \simeq \mathsf{isEquiv}(F)$ B
2. Consequently, $\mathsf{isEquiv}(F)$ is a mere proposition
3. $(C \simeq_{\mathsf{Cat}} C') \simeq (C \cong_{\mathsf{Cat}} C')$

All notions of equality of categories are equivalent

Proof of categorical choice

Theorem

If A is a category and $F : A \to B$ is an f.f.f., then

$$\prod(b : B), \mathsf{isProp}(\sum(a : A_0), F_0 a \cong b)$$

Proof.

Take $(a, f), (a', f')$ of the above type, and observe that $f^{-1} f' : F_0 a' \cong F_0 a$. By f.f.f., there is $g :\equiv F^{-1}_{a',a}(f^{-1} f') : a' \cong a$ and so by category $p :\equiv \mathsf{idtoiso}_C(g) : a' = a$. By transport lemmas, we have $\mathsf{transport}(p, f') = f$ so that $(a', f') = (a, f)$. □

The better choice

Categorical choice is a *theorem*

For any category A and f.f.f. $F : A \to B$

$$\left(\prod_{(b:B)} \left\| \sum_{(a:A)} (Fa \cong b) \right\|\right) \simeq \left\| \sum_{(G:\prod(b:B),A)} \prod_{(b:B)} (FGb \cong b) \right\|$$

Set choice is only *consistent*

For any set B, family $A : B \to \mathsf{Set}$, and $P : \sum(b:B), A \to \mathsf{Prop}$

$$\left(\prod_{(b:B)} \left\| \sum_{(a:A(b))} P(a,b) \right\|\right) \simeq \left\| \sum_{(g:\prod(b:B),A)} \prod_{(b:B)} P(gb,b) \right\|$$

Rezk Completion

The case for preorders $(P, \leq : P \times P \to \mathsf{Prop})$

Let's write $x \cong y$ for $(x \leq y) \times (y \leq x)$, and we know that we generally only have $x = y \to x \cong y$.

In this case we have the ideal completion:

$(- \circ W) : [\overline{P}, Q] \to [P, Q]$ is an isomorphism, $\begin{array}{ccc} P & \xrightarrow{W} & \overline{P} \\ m \downarrow & \swarrow & \overline{m} \\ Q & & \end{array}$

"Universal solution in which $(x = y) \simeq (x \cong y)$"

A translation

Preorder	Precategory
Enriched over 2	Enriched over Set
Ideal	Presheaf
Principal ideal	Representable functor
Order	Category

We know where to look.

The strategy for precategories

We need to *reflect* precats into cats

$$\mathsf{Rezk} : \prod_{(A : \mathsf{Precat})} \sum_{(\overline{A} : \mathsf{Cat})} \sum_{(W : A \to \overline{A})} \mathsf{isWeakEquiv}(W)$$

Then prove: Rezk has the correct universal property

$(- \circ W) : [\overline{A}, C] \to [A, C]$ is an isomorphism,
$$\begin{array}{ccc} A & \xrightarrow{W} & \overline{A} \\ F \downarrow & \swarrow \overline{F} & \\ C & & \end{array}$$

To construct the Rezk completion we must appeal to the Yoneda lemma

This slide is about the Yoneda lemma

You knew it was coming

For any precatgory A, $a : A_0$, and $F : \widehat{A}_0 :\equiv [A, \mathsf{Set}]_0$ there is a term of type $\widehat{A}(\mathsf{y}a, F) \cong F_0 a$ varying naturally in a and F.

Facts

- y is always an f.f.f.
- If A is a category then
  - y_0 is an embedding of types, i.e., $(y_0 a = y_0 b) \simeq (a = b)$
  - For presheaves, $\mathsf{isRep}(F)$ is a mere proposition

Definition

Given a precategory A, let $\overline{A}$ be the full subprecategory of $\widehat{A}$ comprising representables.

As the inclusion of $\overline{A}$ into $\widehat{A}$ is an f.f.f. and an embedding of objects, $\overline{A}$ is a *category*.

Constructing Rezk

The codomain restriction of $\mathsf{y} : A \to \overline{A}$ is f.f. and merely ess. surj., so a weak equivalence.

Further directions in univalent categories

- Redoing the classical theory with categories, many instances of set-theoretic choice
  - Limits, Universal Properties, Monadicity
  - (G/S/F)AFT
  - …
- General versions of the structure identity principle, where does it fail?
- Higher versions, already Cat forms a '2-category' in our sense, strictification theorems

References

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- M. H. Escardó, “A self-contained, brief and complete formulation of Voevodsky’s Univalence Axiom”, arXiv:1803.02294
- The Univalent Foundations Programme, “Homotopy Type Theory: Univalent Foundations of Mathematics”, https://homotopytypetheory.org/book

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