# 2-lessons from Australian Category Theory Mates (and ~~maybe~~ probably not Doctrinal Adjunction)

a poem in free verse for the JHU Category Theory Seminar
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(with different apologies)

Contents:

Recall adjunctions: $L \dashv R$ is ...

**1-dim'l**

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] & \mathcal{D} \arrow[l, bend left, "R"] \end{tikzcd}$$

**2-dim'l**

$$+ \quad \begin{tikzcd} \mathcal{D} \arrow[d, "R"'] & \mathcal{C} \arrow[l, "L"'] \arrow[dl, equal] \arrow[dl, phantom, bend right, "\Nearrow\eta"] \\ \mathcal{C} & \end{tikzcd} \quad + \quad \begin{tikzcd} & \mathcal{D} \arrow[d, "R"] \arrow[dl, equal] \arrow[dl, phantom, bend left, "\Nearrow\varepsilon"'] \\ \mathcal{D} & \mathcal{C} \arrow[l, "L"] \end{tikzcd}$$

unit $\qquad\qquad$ counit

**3-dim'l**

$$+ \quad \begin{tikzcd} L & LRL \arrow[l, Leftarrow, "\varepsilon L"'] \\ & L \arrow[u, Rightarrow, "L\eta"'] \arrow[ul, equal] \arrow[ul, phantom, bend right, "\equiv"] \end{tikzcd} \quad + \quad \begin{tikzcd} R & \\ RLR \arrow[u, Rightarrow, "R\varepsilon"] & R \arrow[l, Leftarrow, "\eta R"] \arrow[ul, equal] \arrow[ul, phantom, bend left, "\equiv"'] \end{tikzcd}$$

triangle equalities

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Thinking about the data in terms of dimensions allows us to liberate ourselves from the complications of dealing with actual categories – after all, we're category theorists!

Recall 2-category: $\quad$ 0-cells $A, B, \dots$ $\quad$ 1-cells $A \xleftarrow{f} B, \dots$

2-cells $\begin{tikzcd} A & B \arrow[l, bend right, "f"'] \arrow[l, bend left, "f'"] \arrow[l, phantom, "\Downarrow\alpha"] \end{tikzcd}, \dots$ $\quad$ identities $A = A$, $\begin{tikzcd} A & A \arrow[l, bend right, "f"'] \arrow[l, bend left, "f"] \arrow[l, phantom, "\parallel"] \end{tikzcd}, \dots$

+ composition + laws

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**Definition**: An adjunction in a 2-category $\mathbb{C}$ is

$$
\begin{tikzcd}
C \arrow[r, bend left, "L"] & D \arrow[l, bend left, "R"]
\end{tikzcd}
\quad + \quad
\begin{tikzcd}
D & C \arrow[l, "L"'] \arrow[dl, equal] \\
C \arrow[from=1-1, "R"'] \arrow[ur, phantom, "\Uparrow\eta" near start] &
\end{tikzcd}
\quad + \quad
\begin{tikzcd}
& D \arrow[d, "R"] \\
D \arrow[ur, equal] \arrow[r, phantom, "\Uparrow\varepsilon" near end] & C \arrow[l, "L"]
\end{tikzcd}
$$

unit          counit

$$
+ \quad
\begin{tikzcd}
L & LRL \arrow[l, Rightarrow, "\varepsilon L"'] \\
& L \arrow[u, Rightarrow, "L\eta"'] \arrow[ul, equal] \arrow[ul, phantom, "\equiv" near start, shift left=3]
\end{tikzcd}
\quad + \quad
\begin{tikzcd}
R \arrow[dr, equal] & \\
RLR \arrow[u, Rightarrow, "R\varepsilon"] & R \arrow[l, Rightarrow, "\eta R"] \arrow[ul, phantom, "\equiv" near end, shift left=3]
\end{tikzcd}
$$

(these pictures can also be understood in a bi-category)

~~Exercises:~~

~~Examples~~:

- An adjunction in $\mathrm{CAT}$ is ... an adjunction
- An adjunction in ~~C~~ a 1-cat viewed as a discrete 2-cat is ... an isomorphism (how low can you go?)
- An adjunction in $\mathrm{REL}$

  $\left\{\begin{array}{ll} \text{objects} & \text{sets} \\ r : A \nrightarrow B & \text{relations } r \subseteq A \times B \\ r \leq s, & \text{inclusion} \end{array}\right.$

  is ... a function $f : A \to B$

  and the relations $\mathrm{graph}(f) \subseteq A \times B$

  $\mathrm{graph}(f)^{o} \subseteq B \times A$

  categories
  profunctors
  nat. transf.

- An adjunction in $\mathrm{Prof}$ is ... almost a functor

What about your favourite 2-dimensional cat?

Now we're free to concentrate on adjunctions-as-objects.
Which begs the question*: what should a morphism of adjunctions be?

* in a non-technical sense

Well...
$$\begin{tikzcd} A \arrow[d] & B \arrow[l, "f"'] \arrow[d] \arrow[dl, phantom, "\Nwarrow\Swarrow"] \\ A' & B' \arrow[l, "f'"] \end{tikzcd} \qquad \text{and or or} \qquad \begin{tikzcd} B \arrow[d] & A \arrow[l, "g"'] \arrow[d] \arrow[dl, phantom, "\Nwarrow\Searrow"] \\ B' & A' \arrow[l, "g'"] \end{tikzcd} \qquad \text{where} \quad \begin{matrix} f \dashv g \\ f' \dashv g' \end{matrix}$$
$$\sigma \qquad\qquad\qquad\qquad \tau$$

subject to?

Let's draw some pictures

$$\begin{tikzcd} B \arrow[d, "k"'] & A \arrow[l, "g"'] \arrow[d, "h"] \arrow[dl, phantom, "\Searrow\tau"] \\ B' & A' \arrow[l, "g'"] \end{tikzcd} + \begin{tikzcd} A \arrow[d, "h"'] & B \arrow[l, "f"'] \arrow[d, "k"] \arrow[dl, phantom, "\Nwarrow\sigma"] \\ A' & B' \arrow[l, "f'"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} & A \arrow[dl, equal] \arrow[d, "g"] \arrow[dl, phantom, "\Nwarrow\varepsilon"] \\ A \arrow[d, "h"'] & B \arrow[l, "f"'] \arrow[d, "k"] \arrow[dl, phantom, "\Nwarrow\sigma"] \\ A' & B' \arrow[l, "f'"] \end{tikzcd} \quad \text{vs} \quad \begin{tikzcd} B \arrow[d, "k"'] & A \arrow[l, "g"'] \arrow[d, "h"] \arrow[dl, phantom, "\Searrow\tau"] \\ B' \arrow[d, "f'"'] & A' \arrow[l, "g'"'] \arrow[dl, equal] \arrow[dl, phantom, "\Searrow\varepsilon'"] \\ A' & \end{tikzcd}$$

so a natural condition of compatibility would be
"commutes with co-units"

$$\begin{tikzcd} A \arrow[d, "h"'] & B \arrow[l, "f"'] \arrow[d, "k"] \arrow[dl, phantom, "\Nwarrow\sigma"] \\ A' & B' \arrow[l, "f'"] \end{tikzcd} + \begin{tikzcd} B \arrow[d, "k"'] & A \arrow[l, "g"'] \arrow[d, "h"] \arrow[dl, phantom, "\Searrow\tau"] \\ B' & A' \arrow[l, "g'"] \end{tikzcd} + \begin{tikzcd} f'g'h \arrow[d, Rightarrow, "\varepsilon' h"'] & f'kg \arrow[l, Rightarrow, "f'\tau"'] \arrow[d, Rightarrow, "\sigma g"] \arrow[dl, phantom, "/\!/\!/"] \\ h & hfg \arrow[l, Rightarrow, "h\varepsilon"] \end{tikzcd}$$

but this is not the only picture we can draw.

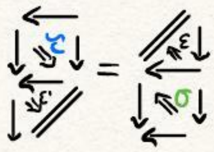
Let's make an odd definition:

<u>Definition</u>: Given a 2-category $\mathbb{C}$ and adjunctions $A \overset{f \dashv g}{\longleftarrow} B$ , $A' \overset{f' \dashv g'}{\longleftarrow} B'$, a **morphism of adjunctions** from $f \dashv g$ to $f' \dashv g'$ is the data

$$\begin{tikzcd} A \arrow[d, "h"'] & B \arrow[l, "f"'] \arrow[d, "k"] \arrow[dl, phantom, "\Nwarrow \sigma"] \\ A' & B' \arrow[l, "f'"] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[d, "k"'] & A \arrow[l, "g"'] \arrow[d, "h"] \arrow[dl, phantom, "\Searrow \tau"] \\ B' & A' \arrow[l, "g'"] \end{tikzcd}$$

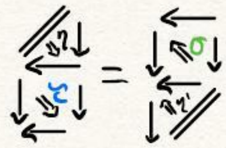
subject to the constraint that at least one of the following equations holds.

$$\begin{tikzcd} f'g'h \arrow[d, Rightarrow, "\varepsilon' h"'] & f'kg \arrow[l, Rightarrow, "f'\tau"'] \arrow[d, Rightarrow, "\sigma g"] \arrow[dl, phantom, "="] \\ h & hfg \arrow[l, Rightarrow, "h\varepsilon"] \end{tikzcd}$$



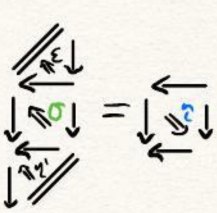
(a)

$$\begin{tikzcd} g'f'k \arrow[d, Rightarrow, "g'\sigma"'] & k \arrow[l, Rightarrow, "\eta' k"'] \arrow[d, Rightarrow, "k\eta"] \arrow[dl, phantom, "="] \\ g'hf & kgf \arrow[l, Rightarrow, "\tau f"] \end{tikzcd}$$



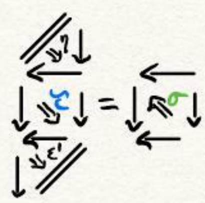
(b)

$$\begin{tikzcd} g'h & kg \arrow[l, Rightarrow, "\tau"'] \arrow[d, Rightarrow, "\eta' kg"] \arrow[dl, phantom, "\equiv"] \\ g'hkf \arrow[u, Rightarrow, "g'h\varepsilon"] & g'f'kg \arrow[l, Rightarrow, "g'\sigma g"] \end{tikzcd}$$



(c)

$$\begin{tikzcd} hf & f'k \arrow[l, Rightarrow, "\sigma"'] \arrow[d, Rightarrow, "f'k\eta"] \arrow[dl, phantom, "\equiv"] \\ f'g'hf \arrow[u, Rightarrow, "\varepsilon' hf"] & f'kgf \arrow[l, Rightarrow, "f'\tau f"] \end{tikzcd}$$



(d)

This is an odd definition for several reasons:

1. ($\underline{a}$) and ($\underline{b}$) are some compatibility conditions, but ($\underline{c}$) and ($\underline{d}$) say that one of $\sigma$ or $\tau$ is not data!

2. What sort of junk is "at least one of"? Can we tell which? Does it matter?

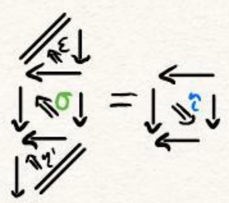
3. We claim these are "morphisms", but how do they compose? What if two composible morphisms satisfy different conditions?

Let's address 2 first:

<u>Fundamental exercise</u> of <u>adjunctions N°1</u>: Conditions $\underline{a}$, $\underline{b}$, $\underline{c}$, and $\underline{d}$ are all equivalent!

this has a bizarre consequence

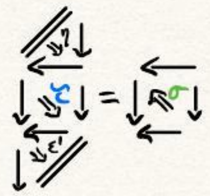
$$
\begin{tikzcd}
g'h & kg \arrow[l, Leftarrow, "\tau"'] \arrow[d, Rightarrow, "\eta' kg"] \\
g'hkf \arrow[u, Rightarrow, "g'h\varepsilon"] \arrow[ur, phantom, "\equiv"] & g'f'kg \arrow[l, Leftarrow, "g'\sigma g"]
\end{tikzcd}
$$



($\underline{c}$)

iff. $\longleftrightarrow$

$$
\begin{tikzcd}
hf & f'k \arrow[l, Leftarrow, "\sigma"'] \arrow[d, Rightarrow, "f'k\eta"] \\
f'g'hf \arrow[u, Rightarrow, "\varepsilon' hf"] \arrow[ur, phantom, "\equiv"] & f'kgf \arrow[l, Leftarrow, "f'\tau f"]
\end{tikzcd}
$$



($\underline{d}$)

$\sigma$ <u>uniquely determines and is uniquely determined by</u> $\tau$

This is category theory, so of course we'll meditate on trivialities, and of course those trivialities will end up having profound consequences.

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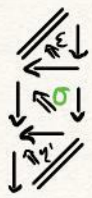
Terminology: given
$$\begin{tikzcd} A \arrow[d, "h"'] \arrow[dr, phantom, "\Nwarrow\sigma"] & B \arrow[l, "f"'] \arrow[d, "k"] \\ A' & B' \arrow[l, "f'"] \end{tikzcd} , \quad \begin{tikzcd} B \arrow[d, "k"'] \arrow[dr, phantom, "\Searrow\tau"] & A \arrow[l, "g"'] \arrow[d, "h"] \\ B' & A' \arrow[l, "g'"] \end{tikzcd}$$
a morphism of adjunctions

we call $\tau$ the mate $m(\sigma)$ of $\sigma$ w.r.t. $f \dashv g$ and $f' \dashv g'$

(by extension $m^{-1}(\tau) = \sigma$)

$$\begin{tikzcd} g'h & kg \arrow[l, Rightarrow, "m(\sigma) = \tau"'] \arrow[d, Rightarrow, "\eta' kg"] \\ g'hkf \arrow[u, Rightarrow, "g'h\varepsilon"] \arrow[ur, phantom, "\equiv"] & g'f'kg \arrow[l, Rightarrow, "g'\sigma g"] \end{tikzcd}$$

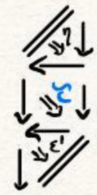
(c)



iff. $\longleftrightarrow$

$$\begin{tikzcd} hf & f'k \arrow[l, Rightarrow, "m^{-1}(\tau) = \sigma"'] \arrow[d, Rightarrow, "f'k\eta"] \\ f'g'hf \arrow[u, Rightarrow, "\varepsilon' hf"] \arrow[ur, phantom, "\equiv"] & f'kgf \arrow[l, Rightarrow, "f'\tau f"] \end{tikzcd}$$

(d)



$\rightarrow$ $m : \mathcal{C}(B,A')(f'k, hf) \xrightarrow{\;m\;} \mathcal{C}(A,B')(kg, g'h)$

is a bijection!

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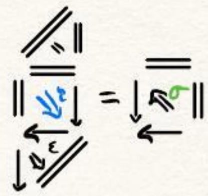
Although "bijection" is a bad word this is already good enough to prove:

**Lemma:** If $(f, g, \eta', \varepsilon)$ and $(f, g, \eta, \varepsilon)$ are adjunctions, then $\eta = \eta'$

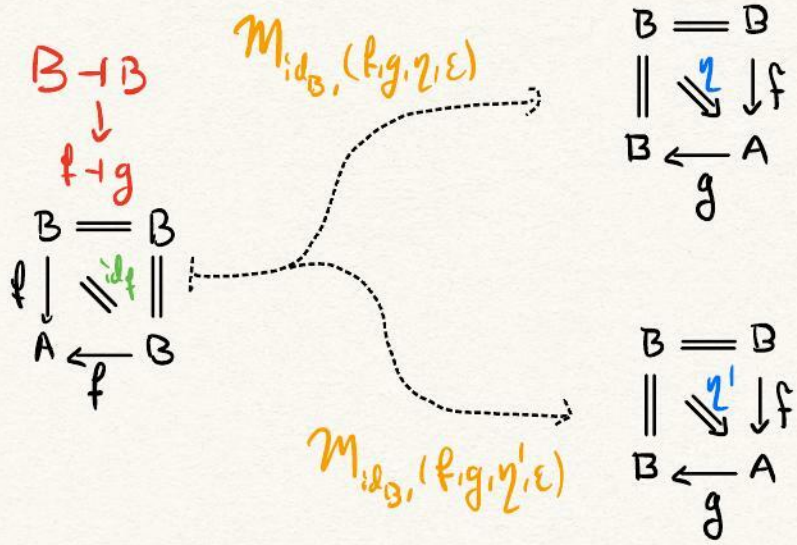
(in particular true for adjoint equivalences w/ rectification)

Proof: 1. $M^{-1}_{id_B, f \dashv g}$ is independent of the unit of $f \dashv g$

$$\begin{tikzcd} B \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & B \arrow[d, "f"] \\ B & A \arrow[l, "g"] \end{tikzcd} \longmapsto \begin{tikzcd} f \arrow[d, phantom] \arrow[dr, phantom, "\equiv"] & f \arrow[l, Leftarrow, "\sigma"'] \arrow[d, equal] \\ fgf \arrow[u, Rightarrow, "\varepsilon f"] & f \arrow[l, Leftarrow, "f\eta"] \end{tikzcd} \rightsquigarrow$$  (d)

Beck-Chevalley condition

2.




∎

Will improve this later.

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Let's return to the outstanding item on our list of odd facets of our definition of morphism of adjunction:

we call them morphisms, but do they compose?
are there identities?

# Fundament exercise of adjunctions №2: ff

$$
\begin{tikzcd}
A \arrow[d, "h"'] \arrow[dr, phantom, "\Nwarrow\sigma"] & B \arrow[l, "f"'] \arrow[d, "k"] \\
A' & B' \arrow[l, "f'"]
\end{tikzcd}
\quad
\begin{tikzcd}
B \arrow[d, "k"'] \arrow[dr, phantom, "\Searrow\tau"] & A \arrow[l, "g"'] \arrow[d, "h"] \\
B' & A' \arrow[l, "g'"]
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
A' \arrow[d, "h'"'] \arrow[dr, phantom, "\Nwarrow\sigma'"] & B' \arrow[l, "f'"'] \arrow[d, "k'"] \\
A'' & B'' \arrow[l, "f''"]
\end{tikzcd}
\quad
\begin{tikzcd}
B' \arrow[d, "k'"'] \arrow[dr, phantom, "\Searrow\tau'"] & A' \arrow[l, "g'"'] \arrow[d, "h'"] \\
B'' & A'' \arrow[l, "g''"]
\end{tikzcd}
$$

are morphisms of adjunctions $(f''\dashv g'') \xleftarrow{(\sigma',\tau')} (f'\dashv g') \xleftarrow{(\sigma,\tau)} (f\dashv g)$ then the pastings

$$
\begin{tikzcd}
A \arrow[d, "h"'] \arrow[dr, phantom, "\Nwarrow\sigma"] & B \arrow[l, "f"'] \arrow[d, "k"] \\
A' \arrow[d, "h'"'] \arrow[dr, phantom, "\Nwarrow\sigma'"] & B' \arrow[l] \arrow[d, "k'"] \\
A'' & B'' \arrow[l, "f''"]
\end{tikzcd}
\quad
\begin{tikzcd}
B \arrow[d, "k"'] \arrow[dr, phantom, "\Searrow\tau"] & A \arrow[l, "g"'] \arrow[d, "h"] \\
B' \arrow[d, "k'"'] \arrow[dr, phantom, "\Searrow\tau'"] & A' \arrow[l] \arrow[d, "h'"] \\
B'' & A'' \arrow[l, "g'"]
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\cdot \arrow[d, "h'h"'] & \cdot \arrow[l, "f"'] \arrow[d, "k'k"] \\
\cdot & \cdot \arrow[l, "f''"]
\end{tikzcd}
\quad
\begin{tikzcd}
\cdot \arrow[d, "k'k"'] & \cdot \arrow[l, "g"'] \arrow[d, "h'h"] \\
\cdot & \cdot \arrow[l, "g''"]
\end{tikzcd}
$$

are a morphism of adjunctions $(f''\dashv g'') \longleftarrow (f\dashv g)$.

//

[This coupled with the observation that

$$
\begin{tikzcd}
A \arrow[d, equal] \arrow[dr, phantom, "\Nwarrow id_f"] & B \arrow[l, "f"'] \arrow[d, equal] \\
A & B \arrow[l, "f"]
\end{tikzcd}
\quad
\begin{tikzcd}
B \arrow[d, equal] \arrow[dr, phantom, "\Searrow id_g"] & A \arrow[l, "g"'] \arrow[d, equal] \\
B & A \arrow[l, "g"]
\end{tikzcd}
$$

means that we can finally derive a category whose objects are adjunctions!

but, err, which?

keeping $\sigma$ and $\tau$ both around is pointless (m) oo

Given a 2-category $\mathbb{C}$ we may define $\sigma\mathrm{Adj}\mathbb{C}$ whose objects are adjunctions $f \dashv g$ and whose morphisms are

$$\begin{tikzcd} A \arrow[d, "h"'] & B \arrow[l, "f"'] \arrow[d, "k"] \arrow[dl, phantom, "\Nwarrow\sigma"] \\ A' & B' \arrow[l, "f'"] \end{tikzcd}$$

<u>OR</u>

Given a 2-category $\mathbb{C}$ we may define $\tau\mathrm{Adj}\mathbb{C}$ whose objects are adjunctions $f \dashv g$ and whose morphisms are

$$\begin{tikzcd} B \arrow[d, "k"'] \arrow[dr, phantom, "\Searrow\tau"] & A \arrow[l, "g"'] \arrow[d, "h"] \\ B' & A' \arrow[l, "g'"] \end{tikzcd}$$

but, uh, these are different right?

Of course not, this is category theory! Either we don't make choices or they are immaterial:

$$\underbrace{\sigma\mathrm{Adj}\mathbb{C}} \qquad\qquad \underbrace{\tau\mathrm{Adj}\mathbb{C}}$$

objects

$$\begin{tikzcd} A \arrow[d, "h"'] & B \arrow[l, "f"'] \arrow[d, "k"] \arrow[dl, phantom, "\Nwarrow\sigma"] & & B \arrow[d, "k"'] \arrow[dr, phantom, "\Searrow m\sigma"] & A \arrow[l, "g"'] \arrow[d, "h"] \\ A' \arrow[d, "h'"'] & B' \arrow[l] \arrow[d, "k'"] \arrow[dl, phantom, "\Nwarrow\sigma'"] & \overset{\text{mate}}{\longmapsto} & B' \arrow[d, "k'"'] \arrow[dr, phantom, "\Searrow m\sigma'"] & A' \arrow[l] \arrow[d, "h'"] \\ A'' & B'' \arrow[l, "f''"] & & B'' & A'' \arrow[l, "g'"] \end{tikzcd}$$

morphisms

compose $\downarrow$ $\qquad\qquad$ $\downarrow$ compose + "Fundamental exercise of adjunctions №2"

$$\begin{tikzcd} A & B \arrow[l, "f"'] \arrow[d, "k'k"] \\ A'' \arrow[from=u, "h'h"'] \arrow[ur, phantom, "\Leftarrow \begin{bmatrix}\sigma \\ \sigma'\end{bmatrix}"] & B'' \arrow[l, "f''"] \end{tikzcd} \quad \xmapsto{\text{mate}} \quad \begin{tikzcd} B \arrow[d, "k"'] & A \arrow[l, "g"'] \arrow[d, "h"] \\ B'' \arrow[ur, phantom, "\Rightarrow"] & A'' \arrow[l, "g''"] \end{tikzcd} \qquad \begin{bmatrix} m\sigma \\ m\sigma' \end{bmatrix} = m \begin{bmatrix} \sigma \\ \sigma' \end{bmatrix}$$

* we also need the later unnamed obs. about id's

⇝ $\mathcal{M} : \sigma\mathrm{Adj}\,\mathbb{C} \longrightarrow \tau\,\mathrm{Adj}\,\mathbb{C}$ is a functor* which is identity on objects and a bijection on homs

⇝ $\mathcal{M}$ is an isomorphism of categories!

//

"bijection of homs $\sigma\mathrm{Adj}\,\mathbb{C}(f \dashv g, f' \dashv g') \xrightarrow{\cong} \tau\,\mathrm{Adj}(f \dashv g, f' \dashv g')$"

to

"isomorphism of categories $\sigma\mathrm{Adj}\,\mathbb{C} \xrightarrow{\cong} \tau\mathrm{Adj}$"

isomorphisms of categories preserve and reflect all structures and propositions

products
monoids
splittings
⋮

universal props
equations of morphs.
⋮

$\updownarrow$

anything expressible in the language of $\sigma\mathrm{Adj}\,\mathbb{C}$.

//

**Lemma:** If $f \dashv g$ is an adjunction in $\mathbb{C}$ then $g$ is unique up to unique isomorphism.

**Proof:**
$$\begin{tikzcd} A & B \arrow[l, "f"'] \\ A \arrow[u, equal] & B' \arrow[l, "f'"] \arrow[u, equal] \arrow[ul, phantom, "\Leftarrow \sigma"] \end{tikzcd}$$
is an isomorphism iff.
$$\begin{tikzcd} B \arrow[d, equal] & A \arrow[l, "g"'] \arrow[d, equal] \arrow[dl, phantom, "\Rightarrow m(\sigma)"] \\ B & A \arrow[l, "g'"] \end{tikzcd}$$
is

because "$\sigma$ is an isomorphism" is expressible in the language of $\sigma\mathrm{Adj}\,\mathbb{C}$. Take $\sigma = \mathrm{id}_f$. $\blacksquare$

$\exists \sigma^{-1} : f \leftarrow f'$
$[\sigma\sigma^{-1} = \mathrm{id}_f$
$\wedge\ \sigma^{-1}\sigma = \mathrm{id}_{f'}]$

//

But wait, there's more! There's one last thing we can do with morphisms of adjunctions, and this might give you ...

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Double Vision

Recall: adjunctions "compose"

$$\begin{tikzcd} A \arrow[r, bend right, "g"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \arrow[r, bend right, "\hat{g}"'] \arrow[r, phantom, "\perp"] & C \arrow[l, bend right, "\hat{f}"'] \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[r, bend right, "\hat{g}g"'] \arrow[r, phantom, "\perp"] & C \arrow[l, bend right, "f\hat{f}"'] \end{tikzcd}$$

but this behaviour is invisible to $\sigma\mathrm{Adj}\mathbb{C}$ because adjunctions are the objects there.

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Fundament exercise of adjunctions №3:

If
$$\begin{tikzcd} A \arrow[d, "h"'] & B \arrow[l, "f"'] \arrow[d, "k"] \arrow[dl, phantom, "\Nwarrow\sigma"] \\ A' & B' \arrow[l, "f'"] \end{tikzcd} \quad \begin{tikzcd} B \arrow[d, "k"'] & A \arrow[l, "g"'] \arrow[d, "h"] \arrow[dl, phantom, "\Searrow\tau"] \\ B' & A' \arrow[l, "g'"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B \arrow[d, "k"'] & C \arrow[l, "\hat{f}"'] \arrow[d, "j"] \arrow[dl, phantom, "\Nwarrow\hat{\sigma}"] \\ B' & C' \arrow[l, "\hat{f}'"] \end{tikzcd} \quad \begin{tikzcd} C \arrow[d, "j"'] & B \arrow[l, "\hat{g}"'] \arrow[d, "k"] \arrow[dl, phantom, "\Searrow\hat{\tau}"] \\ C' & B' \arrow[l, "\hat{g}'"] \end{tikzcd}$$

are morphisms of adjunctions $\begin{matrix} (f' \dashv g') \xleftarrow{(\sigma,\tau)} (f \dashv g) \\ (\hat{f}' \dashv \hat{g}') \xleftarrow{(\hat{\sigma},\hat{\tau})} (\hat{f} \dashv \hat{g}) \end{matrix}$ then the pastings

$$\begin{tikzcd} A \arrow[d, "h"'] & B \arrow[l, "f"'] \arrow[d, "k"] \arrow[dl, phantom, "\Nwarrow\sigma"] & C \arrow[l, "\hat{f}"'] \arrow[d, "j"] \arrow[dl, phantom, "\Nwarrow\hat{\sigma}"] \\ A' & B' \arrow[l, "f'"] & C' \arrow[l, "\hat{f}'"] \end{tikzcd} \qquad \begin{tikzcd} C \arrow[d, "j"'] & B \arrow[l, "\hat{g}"'] \arrow[d, "k"] \arrow[dl, phantom, "\Searrow\hat{\tau}"] & A \arrow[l, "g"'] \arrow[d, "h"] \arrow[dl, phantom, "\Searrow\tau"] \\ C' & B' \arrow[l, "\hat{g}'"] & A' \arrow[l, "g'"] \end{tikzcd}$$

are a morphism of the composed adjunctions

$$(f'\hat{f}' \dashv \hat{g}'g') \longleftarrow (f\hat{f} \dashv \hat{g}g)$$

"But how do we organise this?" you might cry if you've forgotten the abstract or didn't notice my DOUBLY subtle hints.

(IRL this took me a very long time to realise*, but like most things in mathematics, it's obvious in hindsight. It was very rewarding and i'm about to rob you of that.)

*when i was young and naïve years ago - at most one of those has changed

Given a 2-category $\mathbb{C}$ we may define $\sigma\text{-}\mathbb{L}\mathrm{Adj}\,\mathbb{C}$ to be the double category whose:

- objects are those of $\mathbb{C}$
- vertical morphisms are the 1-cells of $\mathbb{C}$
- horizontal morphisms $A \leftarrow\!\!\!+\!\!- B$ are left adjoints $A \xleftarrow{f} B$ of $\mathbb{C}$
- squares are 2-cells

$$\begin{tikzcd} A \arrow[d, "h"'] & B \arrow[l, "f"'] \arrow[d, "k"] \arrow[dl, phantom, "\Nwarrow\sigma"] \\ A' & B' \arrow[l, "f'"] \end{tikzcd}$$

<u>Exercise</u>: make sense of this!

## OR

Given a 2-category $\mathbb{C}$ we may define $\tau\text{-}\mathbb{L}\mathrm{Adj}\,\mathbb{C}$ to be the double category whose:

- objects are those of $\mathbb{C}$
- vertical morphisms are the 1-cells of $\mathbb{C}$
- horizontal morphisms $A \leftarrow\!\!\!+\!\!- B$ are left adjoints $A \xleftarrow{f} B$ of $\mathbb{C}$
- squares are 2-cells

$$\begin{tikzcd} B \arrow[d, "k"'] & A \arrow[l, "g"'] \arrow[d, "h"] \arrow[dl, phantom, "\Downarrow\tau"] \\ B' & A' \arrow[l, "g'"] \end{tikzcd}$$

## OR

Given a 2-category $\mathbb{C}$ we may define $\tau\text{-}\mathbb{R}\mathrm{Adj}\,\mathbb{C}$ to be the double category whose:

- objects are those of $\mathbb{C}$
- vertical morphisms are the 1-cells of $\mathbb{C}$

- horizontal morphisms $B \mathrel{\leftarrow\!\!\!\!\!+} A$ are right adjoints $B \xleftarrow{g} A$ of $\mathbb{C}$

- squares are 2-cells

$$\begin{tikzcd} B & A \arrow[l, "g"'] \arrow[d, "h"] \\ B' \arrow[from=u, "k"'] \arrow[ur, phantom, "\Searrow\tau"] & A' \arrow[l, "g'"] \end{tikzcd}$$

OR

Given a 2-category $\mathbb{C}$ we may define $\sigma$-$\mathbb{R}\mathrm{Adj}\,\mathbb{C}$ to be the double category whose:

- objects are those of $\mathbb{C}$

- vertical morphisms are the 1-cells of $\mathbb{C}$

- horizontal morphisms $B \mathrel{\leftarrow\!\!\!\!\!+} A$ are right adjoints $B \xleftarrow{g} A$ of $\mathbb{C}$

- squares are 2-cells

$$\begin{tikzcd} A \arrow[d, "h"'] & B \arrow[l, "f"'] \arrow[d, "k"] \\ A' \arrow[ur, phantom, "\Nwarrow\sigma"] & B' \arrow[l, "f'"] \end{tikzcd}$$

//

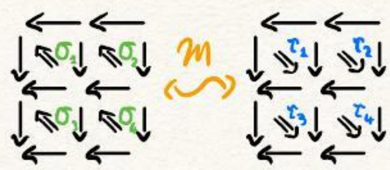
Yet again, of course the choice was immaterial:

Fundament exercises of adjunctions № 1, 2, and 3

$\to$ $\mathcal{M} : \sigma\text{-}\mathbb{L}\mathrm{Adj}\,\mathbb{C} \longrightarrow \tau\text{-}\mathbb{L}\mathrm{Adj}\,\mathbb{C}$

( identity on objects, vertical & horizontal morphisms and mate on squares )

is an isomorphism of double categories




//

"bijection of homs $\sigma\mathrm{Adj}\,\mathbb{C}(f \dashv g, f' \dashv g') \xrightarrow{\cong} \tau\mathrm{Adj}(f \dashv g, f' \dashv g')$"

to

"isomorphism of categories $\sigma\mathrm{Adj}\,\mathbb{C} \xrightarrow{\cong} \tau\mathrm{Adj}$"

to

"isomorphism of double categories $\sigma\text{-}\mathbb{L}\mathrm{Adj}\,\mathbb{C} \xrightarrow{\cong} \tau\text{-}\mathbb{L}\mathrm{Adj}\,\mathbb{C}$"

to

"$\sigma\text{-}\mathbb{L}\mathrm{Adj}\,\mathbb{C} \xrightarrow{\cong} \tau\text{-}\mathbb{L}\mathrm{Adj}\,\mathbb{C} \xrightarrow{\cong} (\tau\text{-}\mathbb{R}\mathrm{Adj}\,\mathbb{C})^{ho^*} \xrightarrow{\cong} (\sigma\text{-}\mathbb{R}\mathrm{Adj}\,\mathbb{C})^{ho^*}$"

* horizontal opposite

†: i'm open to better prepositions here

"anything expressible in the language of $\sigma$-$\mathbb{L}\mathrm{Adj}\,\mathbb{C}$

is preserved* in† and reflected* by† $\tau$-$\mathbb{R}\mathrm{Adj}\,\mathbb{C}$" *with a direction swap

<u>Lemma</u>: Let $\sigma$ be a square in $\sigma$-$\mathbb{L}\mathrm{Adj}\,\mathbb{C}$ where $s, t$ are additionally monads in $\mathbb{C} = \mathbb{V}(\sigma\text{-}\mathbb{L}\mathrm{Adj}\,\mathbb{C})$. Then $(f, \sigma)$ is a colax-morphism of monads iff. $(g, m\sigma)$ is a lax-morphism of monads.

$$
\begin{tikzcd}
A \arrow[d, "{(s,\mu^s,\eta^s)}"'] & B \arrow[l, "f"'] \arrow[d, "{(t,\mu^t,\eta^t)}"] \arrow[dl, phantom, "\Nwarrow\sigma"] \\
A & B \arrow[l, "f"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B \arrow[d, "{(t,\mu^t,\eta^t)}"'] & A \arrow[l, "g"'] \arrow[d, "{(s,\mu^s,\eta^s)}"] \arrow[dl, phantom, "\Searrow m\sigma"] \\
B & A \arrow[l, "g"]
\end{tikzcd}
$$

<u>Proof</u>: The proposition "$(f, \sigma)$ is a colax-morphism of monads" is expressible in the language of vertical + horizontal comps + equalities of squares in $\sigma$-$\mathbb{L}\mathrm{Adj}\,\mathbb{C}$. ▨

$$
\begin{tikzcd}
A \arrow[d, "s"'] & B \arrow[l, "f"'] \arrow[d, "t"] \arrow[dl, phantom, "\Nwarrow\sigma"] \arrow[r, equal] & B \arrow[d, equal] \arrow[dl, phantom, "\Nwarrow\eta^t"] \\
A & B \arrow[l, "f"] \arrow[r, equal] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[d, "s"'] \arrow[r, equal] & A \arrow[d, equal] \arrow[dl, phantom, "\Nwarrow\eta^s"] & B \arrow[l, "f"'] \arrow[d, equal] \arrow[dl, phantom, "\Nwarrow \mathrm{id}_f"] \\
A \arrow[r, equal] & A & B \arrow[l, "f"]
\end{tikzcd}
$$

...

//

similarly

a 2-functor with 2-natural transformations $\mu: T^2 \Rightarrow T$, $\eta: \mathbb{C} \Rightarrow T$

<u>Thm</u> (<u>Doctrinal</u> <u>Adjunction</u>): Let $T$ be a 2-monad on $\mathbb{C}$, and let $\sigma$ be a square in $\sigma$-$\mathbb{L}\mathrm{Adj}\,\mathbb{C}$ where $h, k$ are additionally 2-algebras for $T$. Then $(f, \sigma)$ is a colax-morphism of 2-algebras

$\Updownarrow$

$(g, m\sigma)$ is a lax-morphism of 2-algebras

2-functors preserve adjunctions

$$
\begin{tikzcd}
TA \arrow[d, "h"'] & TB \arrow[l, "Tf"'] \arrow[d, "k"] \arrow[dl, phantom, "\Nwarrow \sigma"] \\
A & B \arrow[l, "f"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
TB \arrow[d, "Tk"'] \arrow[dr, phantom, "\Searrow m\sigma"] & TA \arrow[l, "Tg"'] \arrow[d, "Th"] \\
B & A \arrow[l, "g"]
\end{tikzcd}
$$

<u>Proof</u>: The proposition "$(f, \sigma)$ is a colax-morphism of 2-algebras" is expressible in the language of vertical + horizontal comps + equalities of squares in $\sigma$-$\mathbb{L}\mathrm{Adj}\,\mathbb{C}$.

$$
\begin{tikzcd}
T^2A \arrow[d, "\mu_A"'] & T^2B \arrow[l, "T^2f"'] \arrow[d, "\mu_B"] \arrow[dl, phantom, "\Searrow"] \\
TA \arrow[d, "h"'] & TB \arrow[l, "Tf"'] \arrow[d, "k"] \arrow[dl, phantom, "\Nwarrow \sigma"] \\
A & B \arrow[l, "f"]
\end{tikzcd}
=
\begin{tikzcd}
T^2A \arrow[d, "Th"'] & T^2B \arrow[l, "T^2f"'] \arrow[d, "Tk"] \arrow[dl, phantom, "\Nwarrow T\sigma"] \\
TA \arrow[d, "h"'] & TB \arrow[l, "Tf"'] \arrow[d, "k"] \arrow[dl, phantom, "\Nwarrow \sigma"] \\
A & B \arrow[l, "f"]
\end{tikzcd}
$$

...

//

Actually this proof is not "correct!"

We assumed $\mathbb{M}(T\sigma) = T(\mathbb{M}\sigma)$ and

$$\begin{tikzcd} T^2A \arrow[d, "\mu_A"'] \arrow[dr, phantom, "\mathbin{\rotatebox[origin=c]{45}{$\Leftarrow$}}"] & T^2B \arrow[l, "T^2f"'] \arrow[d, "\mu_B"] \\ TA & TB \arrow[l, "Tf"] \end{tikzcd} \overset{\mathbb{M}}{\longleftrightarrow} \begin{tikzcd} T^2B \arrow[d, "\mu_B"'] \arrow[dr, phantom, "\mathbin{\rotatebox[origin=c]{45}{$\Leftarrow$}}"] & T^2A \arrow[l, "T^2g"'] \arrow[d, "\mu_A"] \\ TB & TA \arrow[l, "Tg"] \end{tikzcd}$$

so our mate theorem is not quite enough
because we said nothing about

2-functors and 2-natural transformations
(eg: $T$) (eg: $\eta, \mu$)

Let's fix this!

//

2-categories
2-functors
2-nat. transf.

double cats
double fun,
vert. dbl-nat transf.

Exercise: 1. $\sigma\text{-}\mathbb{L}\mathrm{Adj}$, $(\tau\text{-}\mathbb{R}\mathrm{Adj})^{ho} : 2\mathrm{Cat} \longrightarrow \mathrm{DblCat}_v$
are both 2-functors.

2. $\mathbb{M} : \sigma\text{-}\mathbb{L}\mathrm{Adj} \overset{\cong}{\Longrightarrow} (\tau\text{-}\mathbb{R}\mathrm{Adj})^{ho}$ is a 2-natural
isomorphism.

"anything expressible in the language of

→ σ-LAdj$\mathbb{C}$, σ-RAdj F, and σ-LAdj α

is preserved* and reflected* τ-$\mathbb{R}$Adj"

*ho

But this still isn't "all imaginable statements"!

$$\begin{tikzcd} A \arrow[r, "f \dashv g"] \arrow[d, "l \dashv r"'] \arrow[dr, phantom, "\sigma"] & B \arrow[d, "l' \dashv r'"] \\ A' \arrow[r, "f' \dashv g'"'] & B' \end{tikzcd} \xrightarrow[\text{adj. ⚠}]{\text{Compose}} \begin{tikzcd} A \arrow[r, "l'f"] \arrow[d, equal] \arrow[dr, phantom, "\sigma"] & B' \arrow[d, equal] \\ A \arrow[r, "f'l"'] & B' \end{tikzcd} \xrightarrow{\text{mate}} \begin{tikzcd} B' \arrow[r, "gr'"] \arrow[d, equal] \arrow[dr, phantom, "\mathfrak{m}_{fl',f'l}\sigma"] & A \arrow[d, equal] \\ B' \arrow[r, "rg'"'] & A \end{tikzcd}$$

$\downarrow$ mate $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ||

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "l'"'] \arrow[dr, phantom, "\mathfrak{m}_{f,f'}\sigma"] & A \arrow[d, "l"] \\ B' \arrow[r, "g'"'] & A' \end{tikzcd} \xrightarrow[\text{mate}]{} \begin{tikzcd} B' \arrow[r, "r'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\mathfrak{m}_{l,l'}\mathfrak{m}_{f,f'}\sigma"] & B \arrow[d, "g"] \\ A' \arrow[r, "r"'] & A \end{tikzcd} \xrightarrow[\text{adj ⚠}]{\text{Compose}} \begin{tikzcd} B' \arrow[r, "gr'"] \arrow[d, equal] \arrow[dr, phantom, "\mathfrak{m}_{l,l'}\mathfrak{m}_{f,f'}\sigma"] & A \arrow[d, equal] \\ B' \arrow[r, "rg'"'] & A \end{tikzcd}$$

these operations don't exist in σ/τ-$\mathbb{L}$/$\mathbb{R}$ Adj!

???