From cocategories to enrichment

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1. Introduction

This document arose from the desire of the author to understand cocategories and the passage from their definition to enrichment. None of what appears here is new(s), but this document aims to give a plain and elementary[1] account of the notion of cocategory and of how the presence of cocategory objects informs the enrichment of the ambient category.

To that end, this document strives to assume only some of the basic theory of categories. In particular it takes for granted the notions of (co)limits and enrichment in a monoidal category, and where feasible all definitions are expanded fully.

2. Cocategory objects

Before we give the definition of cocategory objects let us begin by briefly considering category objects, by way of category objects in Set – that is, (small) categories. A small category essentially comprises the following data: a set of objects, a set of arrows, domain and codomain functions assigning arrows to objects, an identity-arrow function taking objects to arrows, and a composition function taking pairs of composable arrows to a single arrow. This data is subject to the usual properties of being a category, among which are the constraints "the domain of the composite of a pair of arrows is the domain of the first arrow" and "composition is associative".

In reaching for a generalisation of categories to other contexts – category objects in a category $\mathfrak{C}$ – one would go about carefully replacing, in the above description of a category, all mentions of the word "set" with "object of $\mathfrak{C}$", "function" with "arrow of $\mathfrak{C}$", "set of composable pairs of arrows" with a certain pullback, and the properties with commutativity constraints on certain diagrams.

A cocategory object in a category is then the dual of this situation, and so we expect for it to comprise data dual to that of a category object. Although categorical naming would encourage us to attach the prefix "co" to all of the maps, we have no interested in carrying about a map called "cocodomain". Instead, in what follows we have suggestively written what might be called "cocomposition" as "cc", and in place of "coidentity" we have used "pt" (as suggested by our central examples ex. 2.3 and ex. 2.6).

[1] Perhaps not in the technical sense

Now to the definition. There are of course may ways of prescribing what a cocategory object in a category should be – we have 'seen' one already: a cocategory object is a category object in the opposite category. Such phrasings will not do for our purposes and instead we have produced below the most elaborated definition in the hopes that the reader might find it a helpful reference. In particular we have taken care to avoid presuming any structure on the ambient category, and so this definition is in some sense 'maximally local'.

Def. 2.1. Let $\mathfrak{C}$ be a category. A *cocategory object* in $\mathfrak{C}$ comprises the data

- objects I_0, I_1, I_2, and I_3,
- arrows $\overline{0}, \overline{1} : I_0 \to I_1$,
- arrows $i_0, i_1 : I_1 \to I_2$,
- arrows $j_0, j_1 : I_2 \to I_3$,
- an arrow $\mathrm{pt} : I_1 \to I_0$,
- an arrow $\mathrm{cc} : I_1 \to I_2$,

satisfying the following properties.

- The diagrams
$$\begin{array}{ccc} I_0 & \xleftarrow{\mathrm{pt}} & I_1 \\ & \diagdown\!\!\diagdown & \overline{0}\uparrow\uparrow\overline{1} \\ & & I_0 \end{array}, \quad \begin{array}{ccc} I_2 & \xleftarrow{\mathrm{cc}} & I_1 \\ i_0\uparrow & & \uparrow\overline{0} \\ I_1 & \xleftarrow[\overline{0}]{} & I_0 \end{array}, \text{ and } \begin{array}{ccc} I_2 & \xleftarrow{\mathrm{cc}} & I_1 \\ i_1\uparrow & & \uparrow\overline{1} \\ I_1 & \xleftarrow[\overline{1}]{} & I_0 \end{array}$$
are commutative.
- The squares
$$\begin{array}{ccc} I_2 & \xleftarrow{i_1} & I_1 \\ i_0\uparrow & & \uparrow\overline{0} \\ I_1 & \xleftarrow[\overline{1}]{} & I_0 \end{array} \text{ and } \begin{array}{ccc} I_3 & \xleftarrow{j_1} & I_2 \\ j_0\uparrow & & \uparrow i_0 \\ I_2 & \xleftarrow[i_1]{} & I_1 \end{array}$$
are pushouts.
- The diagrams
$$\begin{array}{ccccc} I_1 & \xleftarrow{[\overline{0}\,\mathrm{pt}, \mathrm{id}_{I_2}]} & I_2 & \xrightarrow{[\mathrm{id}_{I_2}, \overline{1}\,\mathrm{pt}]} & I_1 \\ & \diagdown\!\!\diagdown & \uparrow \mathrm{cc} & /\!\!/ & \\ & & I_1 & & \end{array} \text{ and } \begin{array}{ccc} I_3 & \xleftarrow{[j_0 i_0, j_1\,\mathrm{cc}]} & I_2 \\ [j_0\,\mathrm{cc}, j_1 i_1]\uparrow & & \uparrow\mathrm{cc} \\ I_2 & \xleftarrow[\mathrm{cc}]{} & I_1 \end{array}$$
are commutative.

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The first two bullets are sometimes referred to as the *costructural* conditions for a cocategory object, and the final bullet is sometimes referred to as the *counitality* and *coassociativity* conditions. Collectively these conditions form the *cocategory properties*, to which the *cocategory data* is subject. For brevity and readability we will denote a cocategory object $(I_\bullet, i_\bullet, j_\bullet, \overline{0}, \overline{1}, \mathrm{pt}, \mathrm{cc})$ by I.

In a sense to be made clear to the reader in the coming examples, a cocategory may be interpreted as comprising a point (I_0), an arrow (I_1) with two endpoints ($\overline{0}, \overline{1}$), and the structure and properties required to understand the gluing of the arrow with itself in a sensible manner.

Remark 2.2. With some assumptions on the ambient category we might 'present' a cocategory object in $\mathfrak{C}$ by the more familiar diagram shape given below, with the understanding that the other objects and arrows involved in the definition are generated by taking pushouts.

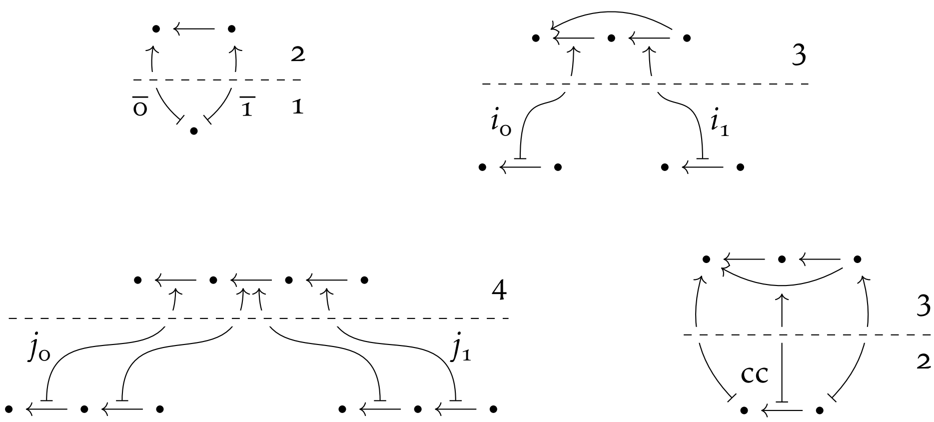
$$I_1 +_{I_0} I_1 \xleftarrow{\ \mathrm{cc}\ } I_1 \underset{\overline{1}}{\overset{\overline{0}}{\leftleftarrows}} \!-\ \mathrm{pt} \rightarrow I_0$$

◁

Example 2.3

Our first example of a cocategory object serves both to cement our intuitions and as a prototypical case – we shall have more to say on this in a moment.

In the category Cat of small categories and functors let 1 denote the terminal category, let 2 denote the free category on an arrow, let 3 denote the free category on a pair of composable arrows, and let 4 denote the free category on a composable triple of arrows. Below we provide the functors $\overline{0}, \overline{1}, i_\bullet, j_\bullet$, and cc schematically and invite the reader to explore the cocategory properties in this pictorial representation. In particular the reader should be careful to note that we have displayed 3 as a pushout $2 +_1 2$ and 4 as a pushout $3 +_2 3$.



In the same way that any category may be constructed from its object, arrows, and composition laws, so too may cocategory objects in Cat be recovered from $(1, 2, 3, 4, \ldots)$ above.

From an object of a category we may seek 'universal' solutions to equipping that object with a cocategory structure, whence our next example. The reader will shortly be able to verify this idea of universality for themself.

Example 2.4

Let C be an object of $\mathfrak{C}$. In what follows we leave the verification of the cocategory properties to the gentle reader.

1. The *discrete cocategory object* C has all objects C and all maps the identity.
2. If there are chosen coproducts $C \coprod C$, $C \coprod C \coprod C$, and $C \coprod C \coprod C \coprod C$ of C with itself in $\mathfrak{C}$, then the *codiscrete cocategory object* C is given by the data:
   (a) $C_i :\equiv \coprod_i C$,
   (b) $\overline{0}, \overline{1}, i_\bullet$, and $j_\bullet$ are given by the coproduct inclusions[2],
   (c) $\mathrm{pt} : C \coprod C \to C$ is given by the codiagonal,
   (d) $\mathrm{cc} : C \coprod C \to C \coprod C \coprod C \coprod C$ is the pairing of the left-most and right-most inclusions of C into $C \coprod C$.

Example 2.5

The (co)discrete cocategory on an object is actually a special case of the following construction. Given a morphism $f : B \to A$ of a category $\mathfrak{C}$ with sufficiently many chosen pullbacks, the cokernel pair of f – the pushout $A +_B A$ of f along itself – canonically gives rise to a cocategory.

Classically this construction is understood to yield an "internal cocongruence relation on A" or sometimes a "coequivalence relation on A" – a cocategory in which the copairing of the pushout inclusions is an epimorphism and $A +_B A$ is internally symmetric. Performing this construction on the initial map $\emptyset \to A$ gives the codiscrete cocategory on A and using the identity instead gives the discrete cocategory.

This final example may be of interest to those with an algebraic-topological bent. While there is no 'natural' setting for cocategories in algebraic topology as the objects at issue are inherently higher dimensional, we may perform the macabre procedure of quotienting by homotopy to arrive at the following example.

Example 2.6

Let Topʜ denote the category comprising general topological spaces and homotopy classes of continuous maps and let $I_n :\equiv [0, n] \subseteq \mathbb{R}$ (note that I_0 is a point).

The maps $\overline{0} :\equiv 0$, $\overline{1} :\equiv 1$, $i_n(t) :\equiv t + n$, and $j_n(t) :\equiv t + n$ exhibit I_3 and I_2 as (up-to-homotopy, so genuine in Topʜ) pushouts $I_2 +_{I_1} I_2$ and $I_1 +_{I_0} I_1$ respectively. The evident map pt and the map $\mathrm{cc}(t) :\equiv 2 * t$ then validate the cocategory properties up to homotopy in Top and so up to equality in Topʜ, and thus I is a cocategory object.

[2] The careful reader may fairly be irked by this statement: we have asked that $C \coprod C \coprod C$ be all of $\coprod_3 C$ and $(C \coprod C) \coprod C$ and $C \coprod (C \coprod C)$. While we trust that a reader so careful is also able to extract our intended meaning, perhaps a still more glorious dawn awaits after which it would take no more work to be precise about such objects and their isomorphisms than would it take to be wrong.

Being the category theorists that we are[3], the only acceptable move after introducing and understanding a new notion is to define an appropriate notion of morphism so that our objects of study themselves might form a category.

Def. 2.7. Let $\mathfrak{C}$ be a category. A morphism $f : I \to I'$ of cocategory objects in $\mathfrak{C}$ comprises arrows $f_0 : I_0 \to I_0'$ and $f_1 : I_1 \to I_1'$ of $\mathfrak{C}$ which render the following diagrams commutative (serially where appropriate).

$$
\begin{array}{ccc}
I_1' & \xleftarrow{\;f_1\;} & I_1 \\
\overline{0}' \uparrow \; \downarrow \mathrm{pt}' \; \uparrow \overline{1}' & & \overline{0} \uparrow \; \downarrow \mathrm{pt} \; \uparrow \overline{1} \\
I_0' & \xleftarrow[\;f_0\;]{} & I_0
\end{array}
\qquad\qquad
\begin{array}{ccc}
I_2' & \xleftarrow{\;f_1 + f_1\;} & I_2 \\
\mathrm{cc}' \uparrow & & \uparrow \mathrm{cc} \\
I_1' & \xleftarrow[\;f_1\;]{} & I_1
\end{array}
$$

⌟

Constr. 2.8. Let $\mathfrak{C}$ be a category. With composition and identities inherited from $\mathfrak{C}$, the collection of cocategory objects and cocategory morphisms in $\mathfrak{C}$ form a category. This is the *category of cocategory objects in* $\mathfrak{C}$, written $\mathrm{cocat}(\mathfrak{C})$. ■

We can now realise the claims about the universality of the constructions given earlier.

Lem. 2.9. *Let $\mathfrak{C}$ be a category, and let $U : \mathrm{cocat}(\mathfrak{C}) \to \mathfrak{C}$ be the functor given on objects as $U(I) :\equiv I_0$ and on morphisms as $U(f_0, f_1) :\equiv f_0$. The assignment on objects $C \mapsto$ "the discrete cocategory on C" extends to a left adjoint to U. If $\mathfrak{C}$ has enough coproducts then the assignment on objects $C \mapsto$ "the codiscrete cocategory on C" extends to a right adjoint to U.* ■

Cor. 2.10. *If $\mathfrak{C}$ has an initial (resp. terminal) object then the discrete cocategory on this object is initial (resp. terminal) in* $\mathrm{cocat}(\mathfrak{C})$.

Remark 2.11. Note that both adjoints to U are fully faithful and that U is faithful, and that $\mathfrak{C}$ may always be recovered up to isomorphism as the subcategory of discrete cocategory objects in $\mathfrak{C}$. In this way, when we later express only a dependence on $\mathrm{ob}\,\mathfrak{C}$ – as opposed to $\mathfrak{C}$ itself – in our arguments, the influence of the categorical structure of $\mathfrak{C}$, the arrows and their compositions, is not lost. ◁

[3] You *are* reading a document entitled "*From cocategories to enrichment*" after all.

3. The structure endowed by cocategory objects

We are now equipped to understand how the category of cocategory objects in a category $\mathfrak{C}$ informs the category of Cat-enrichments on the set of objects $\mathrm{ob}\,\mathfrak{C}$, with respect to aspects of the internal structure of the category $\mathfrak{C}$ itself. See remark 3.10 for a refined version of this statement.

Let us begin with a straightforward lemma.

Lem. 3.1. *If $F : \mathfrak{C}^{\mathrm{op}} \to \mathfrak{D}$ is a pullback preserving functor then F is also a functor on cocategories, $F : \mathrm{cocat}(\mathfrak{C})^{\mathrm{op}} \to \mathrm{cat}(\mathfrak{D})$.* ∎

Without wishing to give the game away just yet, such a lemma is of interest because *every* category is naturally equipped with healthy stock of pullback preserving functors, viz., the representable functors. That is to say,

Observation 3.2. *For every object $C \in \mathrm{ob}\,\mathfrak{C}$, the representable functor at C is a functor $\mathfrak{C}(-, C) : \mathrm{cocat}(\mathfrak{C})^{\mathrm{op}} \to \mathrm{cat}(\mathrm{Set}) \equiv \mathrm{Cat}$. Moreover, this assignment is functorial in C, that is, the Yoneda embedding is a functor $\tilde{y} : \mathrm{cocat}(\mathfrak{C})^{\mathrm{op}} \times \mathfrak{C} \to \mathrm{Cat}$.* ∎

Although this observation might be cause for excitement – we have just shown how to functorially extract, for every object and every cocategory, a category – this is not quite the form of the result we wanted. For that we will need to establish the following.

Observation 3.3. *If $\mathfrak{C}$ has chosen binary products and multiplication preserves pushouts, then product extends to a functor $\times : \mathrm{cocat}(\mathfrak{C}) \times \mathfrak{C} \to \mathrm{cocat}(\mathfrak{C})$. Moreover we have the composite functor $H :\equiv \tilde{y}((\times)^{\mathrm{op}} \times \mathrm{id}_{\mathfrak{C}}) : \mathrm{cocat}(\mathfrak{C})^{\mathrm{op}} \times \mathfrak{C}^{\mathrm{op}} \times \mathfrak{C} \to \mathrm{Cat}$.* ∎

Such a scenario is plausible when $\mathfrak{C}$ is cartesian closed. In this case we may additionally observe that the internal hom functor is also a functor $[-,-] : \mathrm{cocat}\, C^{\mathrm{op}} \times \mathfrak{C} \to \mathrm{cat}(\mathfrak{C})$. That is, exponentiation by a cocategory object yields a category object.

For the next observation we fix some terminology. Let $U : \mathrm{Cat} \to \mathrm{Grph}$ be the forgetful functor and D its left adjoint. For a category $\mathfrak{C}$ we will freely confuse the *set* $\mathrm{ob}\,\mathfrak{C}$ with the *discrete category $DU\mathfrak{C}$*.

Observation 3.4. *We may precompose H with the couint of the adjunction $D \dashv U$ to construct the functor $h :\equiv H(\mathrm{id}_{\mathrm{cocat}(\mathfrak{C})^{\mathrm{op}}} \times \varepsilon_{\mathfrak{C}^{\mathrm{op}}} \times \varepsilon_{\mathfrak{C}}) : \mathrm{cocat}(\mathfrak{C})^{\mathrm{op}} \times \mathrm{ob}\,\mathfrak{C} \times \mathrm{ob}\,\mathfrak{C} \to \mathrm{Cat}$.* ∎

Now the fun begins and we will reveal our surprise: we are going to provide a Cat enrichment of $\mathrm{ob}\,\mathfrak{C}$ predicated on $\mathrm{cocat}(\mathfrak{C})^{\mathrm{op}}$. That is, for each cocategory object in $\mathfrak{C}$ we will functorially assign a 2-category on the objects of $\mathfrak{C}$.

To achieve this we'll use the definition of enrichment involving monoidal categories. That is, let us consider Cat with its cartesian monoidal structure[4] and seek to give a Cat enrichment on the set $\mathrm{ob}\,\mathfrak{C}$, functorial in $\mathrm{cocat}(\mathfrak{C})^{\mathrm{op}}$.

Let us begin by constraining our context to a fixed cocategory object I and seek to construct a Cat-category on objects $\mathrm{ob}\,\mathfrak{C}$, that is, a 2-category[5] with objects $\mathrm{ob}\,\mathfrak{C}$. To fix terminology by $\mathfrak{C}^I$ let us denote the soon-to-be 2-category we shall build from I and $\mathrm{ob}\,\mathfrak{C}$.

[4]There are canonically constructed n-ary products for every n-tuple of categories which are created by the forgetful functor to graphs, in which there are again canonical products.

[5]Observe that the ring of categorical naming schemes has zero divisors for $2\mathrm{Cat} \equiv \mathrm{Cat} - \mathrm{Cat}$.

Described in broad strokes, the objects of $\mathfrak{C}^I$ will be the objects of $\mathfrak{C}$, and the hom-categories $\mathfrak{C}^I(B,A)$ will be the categories $h(I,B,A)$ we have constructed in obs. 3.4. Of course there is a peculiarity of dimension here: we began with a 1-category $\mathfrak{C}$ and we wish to end with a 2-category $\mathfrak{C}^I$ using only $\mathfrak{C}$. By necessity then, the 1- and 2- cells of $\mathfrak{C}^I$ will *both* be *arrows* of $\mathfrak{C}$. Specifically, the objects of $\mathfrak{C}^I(B,A)$ are arrows $B \times I_0 \to A$ and the arrows of $\mathfrak{C}^I(B,A)$ are arrows $B \times I_1 \to A$.

In this way, as the $\mathfrak{C}^I(B,A) :\equiv h(I,A,B)$ are already categories (obs. 3.4), to give the 2-category structure it remains to provide the structural morphisms in Cat and establish some properties. In particular, for every $A,B,C \in \operatorname{ob}\mathfrak{C}$ we must supply functors

$$\circ_{A,B,C} : h(I,C,B) \times h(I,B,A) \to h(I,C,A)$$
$$j_a : 1 \to h(I,A,A)$$

of *horizontal composition* and *horizontal identity*, and show that this family of composition functors is unital with respect to the family of identity functors, and associative.

In what follows we adopt the standard convention of denoting the monoidal product by juxtaposition – $AB :\equiv A \times B$ – but only where we are thinking of a category as a monoidal category. With this in mind, we may elaborate the definitions in our context to see that to realise our construction of $\mathfrak{C}^I$ we we are tasked with the following.

Prob. 3.5 (The 2-category $\mathfrak{C}^I$). *Let $\mathfrak{C}$ be a category with chosen binary products preserving pushouts and let I be a cocategory object in $\mathfrak{C}$. For every $A,B,C,D \in \operatorname{ob}\mathfrak{C}$ and $n \in \{0,1\}$, construct functions of sets*

$$\circ^n_{A,B,C} : \mathfrak{C}(I_n \times B, A) \times \mathfrak{C}(I_n \times C, B) \to \mathfrak{C}(I_n \times A, C)$$

and elements

$$j^n_A \in \mathfrak{C}(I_n \times A, A)$$

which render the following diagrams (serially) commutative

$$\begin{array}{ccc} \mathfrak{C}(I_1 \times C, A) & \xleftarrow{\circ^1_{A,B,C}} & \mathfrak{C}(I_1 \times B, A)\mathfrak{C}(I_1 \times C, B) \\ (\bar{0} \times C)^* \downarrow \;\; \uparrow (\mathrm{pt} \times C)^* \;\; \downarrow (\bar{1} \times C)^* & & (\bar{0} \times B)^*(\bar{0} \times C)^* \downarrow \;\; \uparrow (\mathrm{pt} \times B)^*(\mathrm{pt} \times C)^* \;\; \downarrow (\bar{1} \times B)^*(\bar{1} \times C)^* \\ \mathfrak{C}(I_0 \times C, A) & \xleftarrow[\circ^0_{A,B,C}]{} & \mathfrak{C}(I_0 \times B, A)\mathfrak{C}(I_0 \times C, B) \end{array} \quad (3.6)$$

$$\begin{array}{ccc} \mathfrak{C}(I_1 \times C, A) & \xleftarrow{\circ^1_{A,B,C}} & \mathfrak{C}(I_1 \times B, A)\mathfrak{C}(I_1 \times C, B) \\ (\mathrm{cc} \times C)^* \uparrow & & \uparrow (\mathrm{cc} \times B)^*(\mathrm{cc} \times C)^* \\ \mathfrak{C}(I_2 \times C, A) & \xleftarrow[\gamma_{A,C}(\circ^1_{A,B,C} \times \circ^1_{A,B,C})\nu_{A,B,C}]{} & \mathfrak{C}(I_2 \times B, A)\mathfrak{C}(I_2 \times C, B) \end{array} \quad (3.7)$$

$$\begin{array}{ccc} \mathfrak{C}(I_n \times D, A) & \xleftarrow{\circ^n_{A,C,D}} & \mathfrak{C}(I_n \times C, A)\mathfrak{C}(I_n \times D, C) \\ \circ^n_{A,B,D} \uparrow & & \uparrow (\circ^n_{A,B,C}\mathfrak{C}(I_n \times D, C))\alpha \\ \mathfrak{C}(I_n \times B, A)\mathfrak{C}(I_n \times D, B) & \xleftarrow[\mathfrak{C}(I_n \times D, C)\circ^n_{B,C,D}]{} & \mathfrak{C}(I_n \times B, A)(\mathfrak{C}(I_n \times C, B)\mathfrak{C}(I_n \times D, C)) \end{array} \quad (3.8)$$

where γ and ν are the canonical isomorphisms

$$\begin{aligned}\gamma_{A,C} &: \mathfrak{C}(I_1\times C,A)\times_{\mathfrak{C}(I_0\times C,A)}\mathfrak{C}(I_1\times C,A)\xrightarrow{\cong}\mathfrak{C}(I_2\times C,A)\\ \nu_{A,B,C} &: \mathfrak{C}(I_2\times B,A)\mathfrak{C}(I_2\times C,B)\xrightarrow{\cong}\\ &\quad(\mathfrak{C}(I_1\times B,A)\mathfrak{C}(I_1\times C,B))\times_{\mathfrak{C}(I_0\times B,A)\mathfrak{C}(I_0\times C,B)}(\mathfrak{C}(I_1\times B,A)\mathfrak{C}(I_1\times C,B))\end{aligned}$$

and which satisfy the following equations

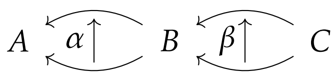
$$(\overline{0}\times A)^*(j_A^1)=j_A^0=(\overline{1}\times A)^*(j_A^1)\quad(\mathrm{pt}\times A)^*(j_A^0)=j_A^1\quad(\mathrm{cc}\times A)^*(\gamma_{A,A}(j_A^1,j_A^1))=j_A^1$$

$$\circ^n_{A,A,B}(j_A,-)=\mathrm{id}_{\mathfrak{C}(I_n\times B,A)}=\circ^n_{A,B,B}(-,j_A)\quad.$$

In the below construction we will make an inspired choice for $\circ^n_{A,B,C}$. Although it may seem that this is perhaps the only sensible choice – and ex. 3.11 later will reassure us that it was indeed – the reader wary of 'inspired' constructions should bear in mind that the historical and natural progression began with ex. 3.11 and its kin before the situation was generalised suitably.

Constr. 3.9 (The 2-category $\mathfrak{C}^I$)**.** We begin by constructing the horizontal identity elements $j_A^n\in\mathfrak{C}(I_n\times A,A)$ for we may dismiss them swiftly: let us set $j_A^n : I_n\times A\to A$ as the projections[6] and then it is straightforward (if tedious in the last case) to verify that the equations hold. We will elaborate the isomorphisms γ and ν later in the construction.

Next we turn to the horizontal composition functions $\circ^n_{A,B,C}$. For fixed $A,B,C\in\mathrm{ob}\,\mathfrak{C}$ we must give a function taking pairs of arrows $(\alpha_n : I_n\times B\to A,\beta_n : I_n\times C\to B)$ to an arrow $\alpha_n\circ^n_{A,B,C}\beta_n : I_n\times C\to A$. A picture of this situation might be the following.



Describing these horizontal composition functions requires some inspired thinking. The construction we give for the value of the function is the composite arrow

$$A\xleftarrow{\alpha_n}I_n\times B\xleftarrow{I_n\times\beta_n}I_n\times(I_n\times C)\xleftarrow{\cong}(I_n\times I_n)\times C\xleftarrow{\Delta}I_n\times C\quad.^7$$

Let us begin to understand this choice by verifying that the properties relating $\circ^n$ to the cocategory structure of I hold. The proofs that this function interacts properly with $\overline{0},\overline{1}$, and pt – the commutativity of diagram 3.6 – are all straightforward and so we have produced only the diagram for $\overline{1}$ below.

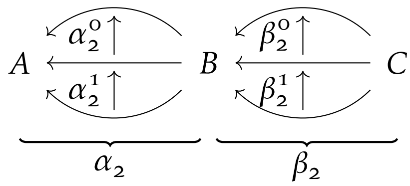
$$\begin{array}{ccccccccc} A & \xleftarrow{\alpha_1} & I_1\times B & \xleftarrow{I_1\times\beta_1} & I_1\times(I_1\times C) & \xleftarrow{\cong} & (I_1\times I_1)\times C & \xleftarrow{\Delta(\overline{1}\times C)} & I_1\times C\\ \| & & \uparrow{\scriptstyle \overline{1}\times B} & & \uparrow{\scriptstyle \overline{1}\times(\overline{1}\times C)} & & \uparrow{\scriptstyle (\overline{1}\times\overline{1})\times C} & & \|\\ A & \xleftarrow[\alpha(\overline{1}\times B)]{} & I_0\times B & \xleftarrow[I_0\times(\beta_1(\overline{1}\times C))]{} & I_0\times(I_0\times C) & \xleftarrow[\cong]{} & (I_0\times I_0)\times C & \xleftarrow[\Delta]{} & I_0\times C\end{array}$$

[6]In this way, the identities are somehow constant in the I_n direction and the identity on A.

[7]This says that at position $t\in I_n$ we do α_n on the result of β_n, both at position t, for every $c\in C$.

By producing these commutative diagrams we have established that horizontal composition respects domains and codomains, and that vertical identity arrows compose horizontally to vertical identity arrows.

The next property which must be shown is the interchange law – the commutativity of diagram 3.7. Let us begin with elements $\alpha_2 \in \mathfrak{C}(I_2 \times B, A)$ and $\beta_2 \in \mathfrak{C}(I_2 \times C, B)$. As we know that I_2 is a pushout $I_1 +_{I_0} I_1$ we may think of α_2 and β_2 as vertically composable pairs of 2-cells, arranged as in the following diagram.



To extract this content we may massage α_2 and β_2 by suitable precompositions. For instance, α_2^0 is the composite

$$A \xleftarrow{\alpha_2} I_2 \times B \xleftarrow{i_0 \times B} I_1 \times B \quad .$$

From this we see that $\alpha_2^0(\overline{1} \times B) = \alpha_2^1(\overline{0} \times B)$ and so on. Said another way, these components are indeed arranged as in the schematic representation above.

To show that diagram 3.7 is commutative is to show that the two evident ways of composing all of the 2-cells pictured above agree. The purpose of ν is thus to break the picture apart and rearrange its constituents so that we might first perform two horizontal compositions on compatible pairs. That is to say, ν factors through the following isomorphisms.

$$\begin{aligned}
&\mathfrak{C}(I_2 \times B, A)\mathfrak{C}(I_2 \times C, B) \cong \mathfrak{C}((I_1 \times B) +_{I_0 \times B} (I_1 \times B), A)\mathfrak{C}((I_1 \times C) +_{I_0 \times C} (I_1 \times C), B) \\
&\cong \Big(\mathfrak{C}(I_1 \times B, A) \times_{\mathfrak{C}(I_0 \times B, A)} \mathfrak{C}(I_1 \times B, A)\Big)\Big(\mathfrak{C}(I_1 \times C, B) \times_{\mathfrak{C}(I_0 \times C, B)} \mathfrak{C}(I_1 \times C, B)\Big) \\
&\cong (\mathfrak{C}(I_1 \times B, A)\mathfrak{C}(I_1 \times C, B)) \times_{\mathfrak{C}(I_0 \times B, A)\mathfrak{C}(I_0 \times C, B)} (\mathfrak{C}(I_1 \times B, A)\mathfrak{C}(I_1 \times C, B))
\end{aligned}$$

Instead of drawing the large commutative diagram whose boundary is diagram 3.7 – though it may interest the reader to produce it for themself – we prove that diagram 3.7 commutes by way of a diagram chase. Of course these are two sides of the same Yoneda.

So, with the terminology established above, we may compute $\nu_{A,B,C}(\alpha_2, \beta_2)$ as the tuple $((\alpha_2^0, \beta_2^0), (\alpha_2^1, \beta_2^1))$ and so begin our diagram chase. Proceeding in this direction we may compute $(\circ^1_{A,B,C} \times \circ^1_{A,B,C})\nu_{A,B,C}(\alpha_2, \beta_2)$ is the pair of composites

$$A \underset{\alpha_2(i_1 \times B)}{\overset{\alpha_2(i_0 \times B)}{\leftleftarrows}} I_1 \times B \underset{I_1 \times (\beta_2(i_1 \times C))}{\overset{I_1 \times (\beta_2(i_0 \times C))}{\leftleftarrows}} I_1 \times (I_1 \times C) \xleftarrow{\cong} (I_1 \times I_1) \times C \xleftarrow{\Delta \times C} I_1 \times C \quad .$$

These composites are of course amenable to rearrangement and we find that the element $(\circ^1_{A,B,C} \times \circ^1_{A,B,C})\nu_{A,B,C}(\alpha_2, \beta_2)$ is equally the following pair of composites.

$$A \xleftarrow{\alpha_2} I_1 \times B \xleftarrow{\beta_2} I_1 \times (I_1 \times C) \xleftarrow{\cong} (I_1 \times I_1) \times C \xleftarrow{\Delta \times C} I_1 \times C \underset{i_1 \times C}{\overset{i_0 \times C}{\leftleftarrows}} I_1 \times C$$

Now we must calculate $\gamma_{A,C}$ of this element. Some inspection reveals that the isomorphism $\gamma_{A,C}$ is given by copairing in this direction, and noting that copairing is functorial and that $[i_0 \times C, i_1 \times C] \equiv \mathrm{id}_{I_2 \times C}$ we see that $\gamma_{A,C}(\circ^1_{A,B,C} \times \circ^1_{A,B,C})\nu_{A,B,C}(\alpha_2, \beta_2)$ is precisely

$$A \xleftarrow{\alpha_2} I_2 \times B \xleftarrow{I_2 \times \beta_2} I_2 \times (I_2 \times C) \xleftarrow{\cong} (I_2 \times I_2) \xleftarrow{\Delta \times C} I_2 \times C \quad .$$

From here it is straightforward to see that the diagram commutes. In this direction our remaining task is only the precomposition of the above with $\mathrm{cc} \times C$,

$$A \xleftarrow{\alpha_2} I_2 \times B \xleftarrow{I_2 \times \beta_2} I_2 \times (I_2 \times C) \xleftarrow{\cong} (I_2 \times I_2) \times C \xleftarrow{\Delta \times C} I_2 \times C \xleftarrow{\mathrm{cc} \times C} I_1 \times C \quad ,$$

and chasing about the diagram along the other composite yields

$$A \xleftarrow{\alpha_2(\mathrm{cc} \times B)} I_1 \times B \xleftarrow{I_2 \times (\beta_2(\mathrm{cc} \times C))} I_1 \times (I_1 \times C) \xleftarrow{\cong} (I_1 \times I_1) \times C \xleftarrow{\Delta \times C} I_1 \times C \quad ,$$

an evidently equal value.

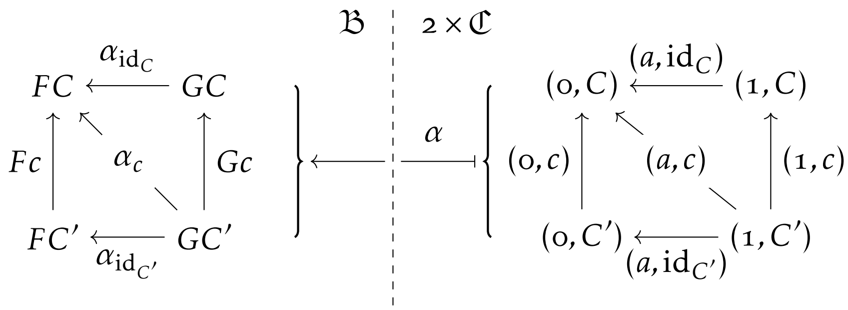
The remaining verification of the associativity property of $\circ^n$ and of the validity of the last equations relating j^n to $\circ^n$ are both straightforward and left to the reader. ■

Remark 3.10. Note that when a cocategory object I has the additional property of I_0 being a terminal object of $\mathfrak{C}$, then we may re-choose our binary products in $\mathfrak{C}$ so that $A \times I_0 \equiv A$ and in so doing, by thereafter enriching using I, we will have succeeded in revealing a 2-category structure living atop the extant 1-categorical structure of $\mathfrak{C}$. That is, $\mathrm{ob}(h(I, A, B)) \equiv \mathfrak{C}(B, A)$ and horizontal composition and identities are the composition and identities of $\mathfrak{C}$. ◁

Example 3.11

The central motivating example is the case of Cat with the cocategory object of ex. 2.3. Here we are in the territory of the above remark, the 2-category structure endowed by our construction on Cat lives precisely over the extant 1-category structure of functors.

Let us examine the details of our prescribed 2-cell structure in $\mathrm{Cat}^{(1,2,3,4,\ldots)}$. Here the 2-cells are functors of the form $\alpha : 2 \times \mathfrak{C} \to \mathfrak{B}$. Should we write F and G for the functors $\alpha(\overline{0} \times \mathfrak{C}), \alpha(\overline{1} \times \mathfrak{C}) : \mathfrak{C} \equiv 1 \times \mathfrak{C} \to \mathfrak{B}$ we find that we may draw the following picture for α on an arrow $c : C' \to C$ of $\mathfrak{C}$.



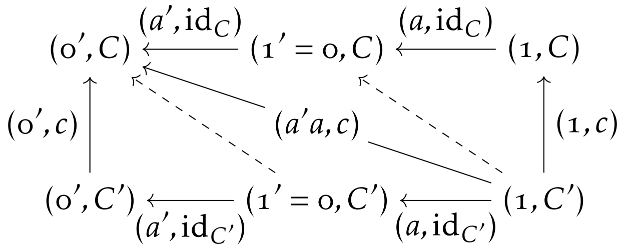
As the diagram on the right is commutative we may infer that its image under α, the diagram on the left, is commutative too. What has emerged then is the (perhaps little-known) definition of a natural transformation as an assignment of *arrows* to *arrows*, obeying a property equivalent to functoriality in $2 \times \mathfrak{C}$. Under the transformation of notation $\mathrm{id}_C \mapsto C$ the left diagram looks far more like the usual definition of a natural transformation, but fundamentally we are still assigning an arrow α_C to the identity arrow of C.

At this point in the example the reader may well feel that such distinctions are petty at best. However, there is a qualitative difference in the nature of vertical and horizontal composition of so-defined natural transformations.

Let us begin with vertical composition so that we might see that it is not quite what one might expect. With the usual definition of natural transformations one simply composes the components of the natural transformations. In our case, however, recall that the vertical composition of compatible 2-cells $\alpha, \beta : 2 \times \mathfrak{C} \to \mathfrak{B}$ is given by the composite

$$\mathfrak{B} \xleftarrow{[\alpha,\beta]} 3 \times \mathfrak{C} \xleftarrow{\mathrm{cc} \times \mathfrak{C}} 2 \times \mathfrak{C} \quad .$$

Beginning with the same commutative square of $2 \times \mathfrak{C}$ above, cocomposition sends (a, c) to the diagonal $(a'a, c)$ of the below commutative rectangle in $3 \times \mathfrak{C}$. From there our definition tells us that the vertical composite on the arrow c is $[\alpha, \beta](a'a, c)$ – and in some sense, we are told nothing else.



In the above diagram the dashed arrows correspond to where we expect α and β to take on their "naturality diagonals" in $\mathfrak{B}$. Of course there is no way to glue these together side-by-side, and so it may be surprising that we can extract a naturality diagonal $(\alpha\beta)_c$ at all.

Recall however that $3 \times \mathfrak{C}$ was displayed as a pushout of $2 \times \mathfrak{C}$ with itself over $1 \times \mathfrak{C}$ by certain functors. That is to say, the universal property states that in order to understand a functor out of $3 \times \mathfrak{C}$ it is enough to understand how it behaves on the two, glued-together copies of $2 \times \mathfrak{C}$ living within it. In practice this is implemented by noticing that the above diagram *is* commutative, and so to extract other arrows of $\mathfrak{B}$ *equal* to $(\alpha\beta)_c$ we may equivalently consider $\alpha_{\mathrm{id}_C}\beta_c$, $\alpha_c\beta_{\mathrm{id}_{C'}}$, or the other two composites involving the functorial direction in $3 \times \mathfrak{C}$. But, of course, we *needn't* compute these other values. In this definition the combinatorics of *all* such computations are subsumed by the fact that $(1, 2, 3, 4, \dots)$ is a cocategory object.

At this point it is hoped that the initially sceptical reader has had their objections allayed and that the dissimilarity of this definition is now at least somewhat evident. But lo dear reader, vertical composition is yet to come.

On the usual account of natural transformations, in order to define horizontal composition one is forced to make a choice of composites of components and functors in order to give the components of the horizontal composite. Of course these choices are *equal*, but that is a theorem and equality does not remove the burden of *specifying* a name. Equality of these composites merely states that our choice didn't matter, that the outcome was the same, but nevertheless we could not have escaped writing *something* down in the first place[8].

In this formulation of naturality no choice is required at all. In fact, horizontal composition follows from functoriality in a rather evident manner. Let $\alpha : 2 \times \mathfrak{C} \to \mathfrak{B}$ and $\beta : 2 \times \mathfrak{D} \to \mathfrak{C}$ be 2-cells, and recall that the horizontal composite is defined as

$$\mathfrak{B} \xleftarrow{\alpha} 2 \times \mathfrak{C} \xleftarrow{2\times\beta} 2 \times (2 \times \mathfrak{D}) \xleftarrow{\cong} (2 \times 2) \times \mathfrak{D} \xleftarrow{\Delta} 2 \times \mathfrak{D} \quad .$$

What does this definition say? Let $d : D' \to D$ be an arrow of $\mathfrak{D}$, our definition of $(\alpha * \beta)_d$ is α_{β_d} – that's it. In more detail, we may trace (a, d) through the diagonal and isomorphism to the arrow $(a, (a, d))$, then to the arrow (a, β_d), and finally to our horizontal composite. With some contemplation, we see that it is the nature of a definition of natural transformation predicated on sending arrows to arrows which allows horizontal composition straightforward: d is an arrow, so β_d is an arrow, so α_{β_d} is an arrow.

Like in the previous case we could calculate that this arrow is equal to any one of the usual components of horizontal composition – a perhaps useful exercise for the curious reader – but like in the previous case we *needn't*. All of this combinatorics is taken care of by asking that $(1, 2, 3, 4, \ldots)$ is a cocategory object and that α and β are functors.

Remark 3.12. From the perspective of this example, one might be tempted to conclude that Cat "knows" about its inherently 2-dimensional nature and so seek interesting co-2-category objects extending $(1, 2, 3, 4, \ldots)$ above. Such a quest will not succeed – there is no "interesting" 3-category structure on Cat given by this form of enrichment –, but it is nevertheless an interesting exercise to understand why. ◂

Remark 3.13. The observation that cocategory objects may be leveraged to induce the 2-categorical structure of Cat was certainly already known by 1967, when J. Bénabou published his monograph "*Introduction to Bicategories*". Specifically on pp. 60-61 of "*Reports of the Midwest Category Seminar*" vol. 47 he writes:

> To construct the 2-dimensional skeleton Bicat[2] of Bicat we use the following idea of category theory:
>
> (i) A functor $h : 2 \times X \to Y$ [...]
>
> [...] such that 2 is a cocategory inside Cat[1] [...].

The author would be grateful for prior references on the genesis of this observation. ◂

[8] Moreover in certain (higher or proof-relevant) contexts what precisely is meant by equality can really muddy the waters here and one wishes to, where possible, avoid such superfluous choices.

Example 3.14

We may check that the naming scheme of ex. 2.4 was appropriately suggestive: (co)discrete cocategory objects give rise to a 1-category with a (co)discrete 2-category structure.

Example 3.15

Our final example concerns the case of this construction in the context of example ex. 2.6, Toph and the topological interval. In this case the construction yields the classical 2-category on topological spaces, continuous maps, and homotopy classes of homotopies between them. This example is why, in certain circles, the construction we have given is known as the "left homotopy enrichment of a category".

At this point we have succeeded in constructing a Cat-enrichment of $\mathrm{ob}\,\mathfrak{C}$ for a fixed cocategory object I, but our stated goal was slightly more general.

Lem. 3.16. *Let $\mathfrak{C}$ be a category satisfying the conditions of prob. 3.5. The assignment on cocategory objects $I \mapsto \mathfrak{C}^I$ of constr. 3.9 extends to a functor $\mathfrak{C}^{(-)} : \mathrm{cocat}(\mathfrak{C})^{\mathrm{op}} \to 2\mathrm{Cat}$.*

Proof. Given a morphism of cocategory objects $f : I' \to I$, construct the 2-functor $\mathfrak{C}^f$ as the identity on objects and with the functorial action of obs. 3.3 on the $h(I,B,A)$. It is straightforward to verify that this commutes appropriately with the horizontal composition and identities and so gives a 2-functor. Functoriality in $\mathrm{cocat}(C)^{\mathrm{op}}$ is immediate. ∎

Finally, should we think of a category $\mathfrak{C}$ as a locally-discrete 2-category then we may reread obs. 3.3 in a slightly different light.

Observation 3.17. *If $\mathfrak{C}$ is a category satisfying the conditions of prob. 3.5, then for each cocategory object I the 2-category $\mathfrak{C}^I$ is a $(\mathfrak{C},\mathfrak{C})$-bimodule. That is, $\mathfrak{C}^I : \mathfrak{C} \times \mathfrak{C}^{\mathrm{op}} \to \mathrm{Cat}$. Moreover this is functorial, $\mathfrak{C}^{(-)} : \mathrm{cocat}(\mathfrak{C})^{\mathrm{op}} \to (\mathfrak{C},\mathfrak{C}) - 2\mathrm{Bimod}$.*

4. Closing thoughts

4.1. Directions of generalisation

This is undoubtedly a neat piece of category theory, but the adventurous reader surely cannot help but wonder about the general principles at play.

Should we reread our construction less carefully, we might be tempted to say that it is over-specialised. There did not appear to be anything special to the theory of cocategories upon which our result depended – beyond the nature of the conclusion of course. Instead, something along the following lines should be true.

Def. 4.1. A sketch $\mathcal{S}$ is a category $\mathcal{S}$ along with a specified class of cocones C and a specified class of cones L on the category. A sketch is a limit sketch when $C = \emptyset$ and dually a colimit sketch when $L = \emptyset$.

A model of a sketch $\mathcal{S}$ in a category $\mathfrak{C}$ is a functor which additionally takes the elements of C and L to colimit and limit cones in $\mathfrak{C}$ respectively. The category of models of a sketch $\mathcal{S}$ in a category $\mathfrak{C}$, denoted $\mathrm{mod}(\mathcal{S},\mathfrak{C})$, has as objects models and as morphisms natural transformations. ⌟

We might instead begin with a *limit* sketch $\mathcal{S}$ and pass to the opposite sketch $\mathcal{S}^{\mathrm{op}}$ – wherein the classes of cones and cocones are interchanged – and consider its models so as to form the category $\mathrm{mod}(\mathcal{S}^{\mathrm{op}}, \mathfrak{C})$ of $\mathcal{S}^{\mathrm{op}}$ models in $\mathfrak{C}$. Should we perform essentially the same construction as before – with the evident caveat that multiplication must now preserve the colimits involved in models as dictated by the class L of $\mathcal{S}^{\mathrm{op}}$ –, the result would be a functor $\mathfrak{C}^{(-)} : \mathrm{mod}(\mathcal{S}^{\mathrm{op}}, \mathfrak{C})^{\mathrm{op}} \to \mathrm{mod}(\mathcal{S}, \mathrm{Set}) - \mathrm{Cat}$.

More still should be true, for we made use of a *particular* functor $\mathfrak{C}^{\mathrm{op}} \times \mathfrak{C}^{\mathrm{op}} \times \mathfrak{C} \to \mathrm{Set}$ which happened to preserve pushouts in its first argument. One could, with little effort, generalise the situation to any functor $F : \mathfrak{C}^{\mathrm{op}} \times \mathfrak{C}^{\mathrm{op}} \times \mathfrak{C} \to \mathfrak{D}$ which interacted appropriately with colimits in its first variable and so obtain a functor $F^{(-)} : \mathrm{mod}(\mathcal{S}^{\mathrm{op}}, \mathfrak{C})^{\mathrm{op}} \to \mathrm{mod}(\mathcal{S}, \mathfrak{D}) - \mathrm{Cat}$ – functorial itself in F – and in so doing consider not only limit sketches.

Perhaps though this is only of passing interest to the reader who wishes to view things from either a monadic or a generalised T-category view. It is to that reader that we must confess we do not have any answers.

It is not presently clear to the author how, given a monad T on $\mathfrak{C}$ dictating, say, the theory of categories, one could mechanically derive a (co)monad dictating the somehow opposite theory but on $\mathfrak{C}$ again and not its opposite. Nor indeed is it clear to the author how one might instead change vantage points and repeat this argument for other flavours of category-like objects – in the simplest case, how might one tell this story beginning with Cat as the category of monads in Span(Set)?

4.2. Further reading

The author wishes to extend their gratitude to S. Awodey for a later discussion on this topic. In particular the author was directed to the following interesting references on the theory of cocategories.

1. "A small observation on co-categories", P. Lefanu Lumsdaine, *Theory and Applications of Categories*, Vol. 25, No. 9, 2011, pp. 247-250.

2. "A characterization of representable intervals", M. A. Warren, *Theory and Applications of Categories*, Vol. 26, No. 8, 2012, pp. 204-232.

The first paper deals with the apparent dearth of somehow 'interesting' cocategory objects in a topos by showing that all cocategory objects in a coherent category are cocongruence relations.

The second paper addresses the question of what, if any, additional algebraically given structure on a cocategory object is equivalent to the finite bicompleteness of the resulting 2-category enrichment. This paper carries out its work in the more general context of a symmetric monoidal closed category with some assumptions – where we have instead relied approximately on the cartesian structure of $\mathfrak{C}$ to perform our enrichment.