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I. Introductory notions

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1. Comma categories and universal arrows

A common theme in category theory is that it is not the objects themselves that are of importance, rather it is the arrows between them which are deserving of study. In this way, one may wonder whether there is an appropriate notion which considers the arrows of a category as the objects of another, new category.

That is, given a category $\mathfrak{C}$ we may wish to form the category $\mathfrak{C}^{\to}$, the arrow category of $\mathfrak{C}$, whose objects are triples $(D, C, f : D \to C)$ with $D, C \in \operatorname{Obj}\mathfrak{C}$ and f an arrow in $\mathfrak{C}$. However, at first glance it may not be entirely clear what the arrows "between arrows" should be.

Should we think harder about this notion, we may suspect that an arrow in $\mathfrak{C}^{\to}$ between (D, C, f) and (D', C', f') would be a pair $(d : D \to D', c : C \to C')$. However, this would give us two, not generally equivalent ways of reaching C, viz., cf and $f'd$. The most obvious requirement then, would be that they must be equal, rendering $\mathfrak{C}^{\to}$ as the category whose objects are arrows in $\mathfrak{C}$ and whose morphisms are commuting squares. Indeed, we choose this definition and generalise the underlying notion to the following.

Def. (I) 1.0.1. Given categories $\mathfrak{C}$ and $\mathfrak{D}$, we may define the functor $\Delta : \mathfrak{C} \to [\mathfrak{D}, \mathfrak{C}]$ through $(\Delta C)D = C$ on objects and $(\Delta f)_D = f$ on morphisms.

Def. (I) 1.0.2. Given three categories $\mathfrak{A}$,$\mathfrak{B}$, and $\mathfrak{C}$, and two functors $F : \mathfrak{A} \to \mathfrak{B}$ and $G : \mathfrak{C} \to \mathfrak{B}$, we define the comma category $(F \downarrow G)$ to be the category whose objects are triples $(A, C, f : FA \to GC)$ and whose morphisms are pairs $(a, c) \in \operatorname{Mor}\mathfrak{A} \times \operatorname{Mor}\mathfrak{C}$ making the following diagram commute.

$$
\begin{array}{ccccccc}
A & \qquad & FA & \xrightarrow{f} & GC & \qquad & C \\
{\scriptstyle a}\downarrow & & {\scriptstyle Fa}\downarrow & & \downarrow{\scriptstyle Gc} & & \downarrow{\scriptstyle c} \\
A' & & FA' & \xrightarrow[f']{} & GC' & & C'
\end{array}
$$

We extend this notation in several ways. First, $(F \downarrow B)$ for $B \in \mathfrak{B}$, wherein it is understood to mean $(F \downarrow \Delta B)$. Alternatively, this may also be interpreted as meaning that $\mathfrak{C} = \mathbb{1}$ and $G\star = B$. Dually, the same extension is applied to $(B \downarrow G)$.

Next, $(\mathfrak{B} \downarrow G)$ and $(F \downarrow \mathfrak{B})$ are understood to be the cases when $\mathfrak{A} = \mathfrak{B}, F = \mathrm{id}_{\mathfrak{B}}$ and $\mathfrak{C} = \mathfrak{B}, G = \mathrm{id}_{\mathfrak{B}}$ respectively. Finally, these extension and the previous can be combined into the cases $(\mathfrak{B} \downarrow B)$ and $(B \downarrow \mathfrak{B})$, called the *over* (sometimes slice) and *under* (coslice) categories respectively.

Remark (I) 1.0.3. A pleasant, *post ex facto*, justification for the requirement of the commuting diagram in the definition of comma categories may be found in the case of natural transforms. Consider $(F \downarrow G)$ where $\mathfrak{A} = \mathfrak{C}$ for two functors $F, G : \mathfrak{C} \rightrightarrows \mathfrak{B}$. Suppose that there was an assignment of objects $\alpha : \mathfrak{B} \to (F \downarrow G)$ such that under the projections of the triple components $\pi_0 : (F \downarrow G) \to \mathfrak{B}$ and $\pi_1 : (F \downarrow G) \to \mathfrak{B}$, $\pi_0\alpha = \pi_1\alpha = \mathrm{id}_{\mathfrak{B}}$. Thus, α would be nothing more than an assignment of arrows $f_B : FB \to GB$ to objects of $\mathfrak{B}$, such that the above diagram commuted in all cases. The careful reader may notice that this is simply the definition of a natural transform. ◃

The language of comma categories also gives us a convenient medium to elucidate a notion of central importance in the theory, viz., limits and colimits.

Prop. (I) 1.0.4. *The limit of a functor $F : \mathfrak{B} \to \mathfrak{C}$ is a terminal object in $(\Delta \downarrow F)$. Dually, the colimit of a functor $F : \mathfrak{B} \to \mathfrak{C}$ is an initial object in $(F \downarrow \Delta)$.*

Proof. We first consider the nature of $(\Delta \downarrow F)$. The categories under consideration here are $\mathfrak{C}$ and $[\mathfrak{B}, \mathfrak{C}]$, with the functor $\Delta : \mathfrak{C} \to [\mathfrak{B}, \mathfrak{C}]$. As such, objects in $(\Delta \downarrow F)$ are pairs $(C, \alpha : \Delta C \to F)$ for $C \in \operatorname{Obj} \mathfrak{C}$ and α a natural transform (elaborated below, right), and morphisms are arrows $c : C \to C'$ in $\mathfrak{C}$ such that the left diagram below commutes.

$$
\begin{array}{c} C \\ {\scriptstyle c}\big\downarrow \\ C' \end{array}
\qquad
\begin{array}{ccc} \Delta C & \xrightarrow{\alpha} & F \\ {\scriptstyle \Delta c}\big\downarrow & \nearrow_{\alpha'} & \\ \Delta C' & & \end{array}
\qquad
\begin{array}{ccc} C & \xrightarrow{\alpha_B} & FB \\ & {\scriptstyle \alpha_{B'}}\searrow & \big\downarrow{\scriptstyle Fb} \\ & & FB' \end{array}
\qquad
\begin{array}{c} B \\ \big\downarrow{\scriptstyle b} \\ B' \end{array}
$$

It is plain to see then that any object (C, α) is a cone over F and that the terminal such object, should it exist, is then the limit $(\lim F, \lambda)$.

$$
\begin{array}{c} C \\ {\scriptstyle c}\,\vdots\downarrow \\ \lim F \end{array}
\qquad
\begin{array}{ccc} \Delta C & \xrightarrow{\alpha} & F \\ {\scriptstyle \Delta c}\big\downarrow & \nearrow_{\lambda} & \\ \Delta \lim F & & \end{array}
$$

The colimit result is achieved dually. ■

Should we examine our choice of notation for the limit of a functor, $\lim F$, we may suspect that this notation belies the true nature of the assignment $F \mapsto \lim F$; a *functor*. Indeed, under the appropriate circumstances and viewed correctly, both lim and colim are functors. Not wishing to become lost in the specifics of limits or colimits, or specialise our arguments more than necessary, we first make an important generalisation before returning to this idea.

Def. (I) 1.0.5. Given a functor $L : \mathfrak{C} \to \mathfrak{D}$, a universal arrow from L to $D \in \operatorname{Obj} \mathfrak{D}$ is a terminal object in $(L \downarrow D)$. Dually, given a functor $R : \mathfrak{D} \to \mathfrak{C}$, a couniversal arrow from $C \in \operatorname{Obj} \mathfrak{C}$ to R is an initial object in $(C \downarrow R)$.

Obviously, this would not be a generalisation of limits and colimits were they not to be examples. In particular, assuming existence

Example (I) 1.0.6

> The limit of a functor F is simply the universal arrow from Δ to F, and dually the colimit of F is the couniversal arrow from F to Δ.

Having made this generalisation, we are inspired to assume the equivalent of a category having all limits of a particular shape, so as to attempt to espy the functorial nature of (co)limits. What follows is a *very* general result along these lines, and the reader may be surprised to find out just how much is a consequence of a seemingly innocuous assumption. We only expressly discuss 'one-half' of this result, as it were, as the rest follows by dualisation.

Prop. (I) 1.0.7. *Given a functor $L : \mathfrak{C} \to \mathfrak{D}$, if every object $D \in \mathrm{Obj}\,\mathfrak{D}$ has a universal arrow $(RD, \varepsilon_D : LRD \to D)$ from L to D, then the following hold*

1. *The assignment $D \mapsto RD$ extends a functor $R : \mathfrak{D} \to \mathfrak{C}$*
2. *The family of arrows $\varepsilon_D : LRD \to D$ determines a natural transform $\varepsilon : LR \to \mathrm{id}_{\mathfrak{D}}$.*
3. *There is a couniversal arrow $(LC, \eta_C : C \to RLC)$ from C to R, for every $C \in \mathrm{Obj}\,\mathfrak{C}$.*
4. *The family of arrows $\eta_C : C \to RLC$ determines a natural transform $\eta : \mathrm{id}_{\mathfrak{C}} \to RL$.*
5. *The following identities hold*
   (a) $(\varepsilon L)(L\eta) = \mathrm{id}_L$
   (b) $(R\varepsilon)(\eta R) = \mathrm{id}_R$

Proof. For (1), the action of R on objects is already defined, we need only extend it to morphisms and demonstrate its functorial nature there. To do so, consider an arrow $f : D \to D'$ in $\mathfrak{D}$, and let (RD, ε_D) and $(RD', \varepsilon_{D'})$ be the universal arrows of D and D' respectively. Observe that $(RD, f\varepsilon_D)$ is an object in $(L \downarrow D')$ and consequently the following diagram commutes.

$$\begin{array}{ccccc} RD & & LRD & \xrightarrow{\varepsilon_D} & D \\ {\scriptstyle Rf}\,\downarrow & & {\scriptstyle LRf}\,\downarrow & & \downarrow\,{\scriptstyle f} \\ RD' & & LRD' & \xrightarrow[\varepsilon_{D'}]{} & D' \end{array}$$

By uniqueness of the arrow Rf it is evident that if $f = gh$ then $RgRh = Rf$ and if $f = \mathrm{id}_D$ then $R\mathrm{id}_D = \mathrm{id}_{RD}$, thus concluding (1). Furthermore, the diagram above is precisely the naturality square for ε, completing the proof of (2).

Next, take $C \in \mathrm{Obj}\,\mathfrak{C}$ and consider the universal arrow $(RLC, \varepsilon_{LC} : LRLC \to LC)$ from L to LC. Observe that (C, id_{LC}) is an object in $(L \downarrow LC)$ and so there must be a unique arrow $\eta_C : C \to RLC$ which has the property $\varepsilon_{LC} L\eta_C = \mathrm{id}_{LC}$, giving (5$a$). Before we prove that (LC, η_C) is the couniversal arrow from C to R, we must demonstrate (4) and (5b).

To see that (4) is the case, consider an arrow $f : C \to C'$ and the universal arrow from L to LC', $(RLC', \varepsilon_{LC'})$ and note that (C, Lf) is an object in $(L \downarrow LC')$. Consequently, there is a unique arrow $u : C \to RLC'$ with $\varepsilon_{LC'} Lu = Lf$, but we have two potential candidates for u, viz., $\eta_{C'} f$ and $RLf\eta_C$.

We check first that $\eta_{C'} f$ is a candidate. Observe that $\varepsilon_{C'} L(\eta_{C'} f) = \varepsilon_{LC'} L\eta_{C'} Lf = Lf$ by (5a). In the case of the second, $\varepsilon_{LC'} LRLf L\eta_C = Lf \varepsilon_{LC} L\eta_C = Lf$ by naturality of ε (2) and then (5a). Consequently, $RLf \eta_C = \eta_{C'} f$ and η is natural, giving (4).

In order to show (5b), we fix $D \in \mathrm{Obj} D$ and consider the universal arrow (RD, ε_D) from L to D. By the terminal property, there is a unique morphism $u : RD \to RD$ with $\varepsilon_D Lu = \varepsilon_D$. Again, however, we have two candidates. The first is obviously id_{RD} while the second is $R\varepsilon_D \eta_{RD}$. To see that the second is suitable, consider that $\varepsilon_D LR\varepsilon_D L\eta_{RD} = \varepsilon_D \varepsilon_{LRD} L\eta_{RD} = \varepsilon_D$ by the naturality of ε (2) and (5a). Consequently, $R\varepsilon_D \eta_{RD} = \mathrm{id}_{RD}$, completing (5).

We are finally in a position to show that (LC, η_C) is the couniversal arrow from C to R. Suppose that there was a $(D, f : C \to RD)$ in $(C \downarrow R)$. We wish to show that there is a unique arrow $u : LC \to D$ making the below diagram commute.

$$\begin{array}{ccccc} C & \xrightarrow{f} & RD & & D \\ & \searrow & \uparrow Ru & & \vdots\uparrow u \\ \eta_C & & RLC & & LC \end{array}$$

The most immediate definition that we can give for u is $u = \varepsilon_D Lf$, where (RD, ε_D) is the universal arrow from L to D, but it we must first convince ourselves that the diagram commutes for this choice of u. Consider that $Ru\eta_C = R\varepsilon_D RLf \eta_C = R\varepsilon_D \eta_{RD} f = f$ by the naturality of η (4) and (5b). It remains to be shown that u is unique.

Suppose there was an arrow $v : LC \to D$ such that $Rv\eta_C = f$, then (C, v) would be an object in $(L \downarrow D)$ and consequently there would have to be a unique morphism $w : C \to RD$ such that $\varepsilon_D Lw = v$, as (RD, ε_D) is terminal by definition. As such, $R\varepsilon_D RLw = Rv$ and so $R\varepsilon_D RLw\eta_C = Rv\eta_C = f$, by the property of v. However, $R\varepsilon_D RLw\eta_C = R\varepsilon_D \eta_{RD} w = w$ by the naturality of η (4) and (5b). Consequently, $w = f$ and so $v = \varepsilon_D Lw = \varepsilon_D Lf = u$, completing the proof. ■

Despite its broad generality, the above result forms merely one of many equivalent ways of stating an even more general relationship between functors, far beyond the notion of universal arrows and even more permeating. In fact, there is more to the arrangement of functors above than meets the eye.

It is clear that despite having natural transforms from LR and RL to the respective identity functors, this setup does not yield an isomorphism of categories in general. In fact, in general, we will not even retrieve an *equivalence* of categories. All is not lost, however, for there *is* a relationship between these functors, one more general than equivalence and isomorhpism, and one whose general form is ubiquitous in the theory. We refer, of course, to the notion of *adjunction* – a topic we shall explore in section 3.

With all of this wonderful generality, it seems almost a shame to have to confine ourselves to the special case of limits and colimits. Nevertheless, in such a case prop. (I) 1.0.7 implies the following.

Cor. (I) 1.0.8. *If* $\mathfrak{C}$ *has all limits of shape* $\mathfrak{B}$*, then* $\lim : [\mathfrak{B}, \mathfrak{C}] \to \mathfrak{C}$ *is a functor.*

The final point to be addressed here is that we may contrive a way to *exchange* limits for colimits (and visa versa) by way of comma categories, that is, we may canonically express one in terms of the other. Unfortunately this re-expression is generally not of interest in that it the colimit formulation of even a finite limit is, in general, taken over a very large category.

Prop. (I) 1.0.9. *A functor $F : \mathfrak{B} \to \mathfrak{C}$ has a limit iff the canonical projection $P : (\Delta \downarrow F) \to \mathfrak{C}$ has a colimit, where $\Delta : \mathfrak{C} \to [\mathfrak{B}, \mathfrak{C}]$ is the constant functor functor.*

Proof. ■

2. The Yoneda lemma and its consequences

2.1. Representable functors

Now that we have an understanding of comma categories and couniversal arrows, we begin by considering what appears to be a special and largely trivial case of such an arrangement.

Recall that there is a bijection of sets $\textsc{Set}(\{*\}, X) \cong X$ (as it happens, natural in X) and so we freely abuse notation and confuse set functions $x : \{*\} \to X$ with points $x \in X$. With this in mind, it stands to reason then that when we have couniversal arrows from $\{*\}$ to functors we may be able to rephrase the situation in terms of specific elements of sets. This is captured in the below, trivial proposition.

Prop. (I) 2.1.1. *Let $F : \mathfrak{C} \to \textsc{Set}$ be a functor. The pair (C, x) is a couniversal arrow from $\{*\}$ to F whenn, for every pair $(C' \in \operatorname{Obj}\mathfrak{C}, x' \in FC')$, there exists a unique $c : C \to C'$ with $Fcx = x'$.*

Although this seems entirely specific to couniversal arrows from $\{*\}$ to functors, under rather meagre assumptions the essence of this characterisation actually subsumes the notion of couniversal arrows. In order to make this claim precise, we first define make a small definition.

Def. (I) 2.1.2. Given a functor $F : \mathfrak{C} \to \textsc{Set}$, a couniversal element of F is a pair (C, x) with $C \in \operatorname{Obj}\mathfrak{C}$ and $x \in FC$ such that for every pair (C', x') with $C' \in \operatorname{Obj}\mathfrak{C}$ and $x' \in FC'$ there exists a unique arrow $c : C \to C'$ with $Fcx = x'$.

Example (I) 2.1.3

Let $F : \textsc{Top}^{\text{op}} \to \textsc{Set}$ be assignment $F(X, \tau) = \tau$ on objects and $F(f^{\text{op}}) = f^{\leftharpoonup}$ on morphisms. It may be checked that this assignment is functorial and that the Sierpinski space with open point 1, $(\mathcal{S}, \{1\})$, is a couniversal element for F.

The key observation here is that, while for general functors we cannot speak of elements of objects in their images, if a functor is somehow uniquely equipped an arrow we can speak of the arrow as an element of a morphism set (assuming sufficient smallness), which is itself the image of an object under a functor.

Prop. (I) 2.1.4. *Let $\mathfrak{C}$ be locally small and $F : \mathfrak{D} \to \mathfrak{C}$ be a functor, then (D, f) is a couniversal arrow from C to F iff $(D, f \in \mathfrak{C}(C, FD))$ is a couniversal element of $\mathfrak{C}(C, F-)$.*

Proof. This follows simply from the definitions. ∎

Example (I) 2.1.5

In the case of $\mathfrak{C} = \textsc{Set}$ and $C = \{*\}$, noting that $\textsc{Set}(\{*\}, X) \cong X$, prop. (I) 2.1.1 is a special case of prop. (I) 2.1.4.

Example (I) 2.1.6

Recall that colimits were couniversal arrows from F to Δ and so we may apply prop. (I) 2.1.4 to find that for $F : \mathfrak{B} \to \mathfrak{C}$, assuming smallness and existence, $(\operatorname{colim} F, \alpha)$ is a couniversal element of $[\mathfrak{B}, \mathfrak{C}](F, \Delta-)$ where α is the cocone.

Now that we have seen that couniversal arrows may be interpreted as specific points of sets with a universal property concerned with set functions, it would seem natural to ask whether there is a yet another phrasing of couniversal arrows which relies on sets and set functions but is not phrased explicitly in terms of 'special' points.

Prop. (I) 2.1.7. *Let $\mathfrak{C}, \mathfrak{D}$ be locally small and $F : \mathfrak{D} \to \mathfrak{C}$ a functor. (D, f) is a couniversal arrow from C to F iff there is a natural isomorphism of functors $\mathfrak{D}(D, -) \cong \mathfrak{C}(C, F-)$.*

Proof. Assume that $(D \in \mathrm{Obj}\,\mathfrak{D}, f : C \to FD)$ is a couniversal arrow from C to F and define $\alpha_{-} : \mathfrak{D}(D, -) \to \mathfrak{C}(C, F-)$ as $\alpha_{D'} u = Fuf$ and $\beta_{-} : \mathfrak{C}(C, F-) \to \mathfrak{D}(D, -)$ as $\beta_{D'} f' = u$ where $u : D \to D'$ is the unique arrow arising from the couniversal property of (D, f). First we check that α, β are isomorphisms and then that they are natural.

Consider $\beta_{D'} \alpha_{D'} u = \beta_{D'}(Ffu)$ and so by the initiality of (D, f), $\beta_{D'}(Ffu) = u$. For the other composite, $\alpha_{D'} \beta_{D'} f' = \alpha_{D'} u$ where $f' = Fuf$ so that $\alpha_{D'} u = Fuf = f'$. For naturality, let $d : D' \to D''$. To see that α is natural observe that we require $\mathfrak{C}(C, Fd)\alpha_{D'} = \alpha_{D''}\mathfrak{D}(D, d)$ but this is immediate as, tracing the action for $u \in \mathfrak{D}(D, D')$ we have $\mathfrak{C}(C, Fd)\alpha_{D'} u = FdFuf$ and $\alpha_{D''}\mathfrak{D}(D, d) = F(du)f$. A simple universal property argument shows that β is natural.

Conversely, assume that $\beta : \mathfrak{C}(C, F-) \to \mathfrak{D}(D, -)$ is a natural isomorphism. We wish to construct an initial object in $(C \downarrow F)$ so we must find an object of $\mathfrak{D}$ and an arrow which satisfy the required properties. As we have already selected $D \in \mathrm{Obj}\mathfrak{D}$ to base β and we always have $\mathrm{id}_D \in \mathfrak{D}(D, D)$ we let $f = {\beta^{-1}}_D \mathrm{id}_D$ and show that (D, f) is initial.

Let $(D', f') \in \mathrm{Obj}(C \downarrow F)$ and observe that we have *an* arrow $u = \beta_{D'} f' : D \to D'$, but to conclude the proof we must establish that this is the unique arrow satisfying $Fuf = f'$. However, this is almost immediate – by naturality and small manipulations we have that $Fuf = \mathfrak{C}(C, Fu){\beta^{-1}}_{D'} \mathrm{id}_D = {\beta^{-1}}_{D'}\mathfrak{D}(D, u)\mathrm{id}_D = {\beta^{-1}}_{D'} u = f'$. If there was another $v : D \to D'$ with $Fvf = f'$ then by the same argument $f' = {\beta^{-1}}_{D'} v$ and so $v = u$. ■

There is actually an added, subtle conclusion which is realised via this proof. Observe that, beginning with a couniversal arrow (D, f) we can move to a natural isomorphism α and then *back* to a couniversal arrow $(D, \bar{f})$. We did not explicitly show that $\bar{f} = f$, but unwinding definitions we find $\bar{f} = {\beta^{-1}}_D \mathrm{id}_D = \alpha_D \mathrm{id}_D = F\mathrm{id}_D f = f$. In fact, a similar statement is true for the reverse process.

If we begin with a natural isomorphism $\alpha : \mathfrak{D}(D, -) \to \mathfrak{C}(C, F-)$, move to a couniversal arrow (D, f) and construct a natural isomorphism $\gamma : \mathfrak{D}(D, -) \to \mathfrak{C}(C, F-)$ it is easy to check that $\alpha = \gamma$. Observe that $\gamma_{D'} u = Fuf$ where $f = \alpha_{D'} \mathrm{id}_D$ so that $\gamma_{D'} u = Fu\alpha_{D'} \mathrm{id}_D = \alpha_{D'} u$ by naturality. All of this is to say that we have shown the stronger claim

Cor. (I) 2.1.8. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be locally small categories and $F : \mathfrak{D} \to \mathfrak{C}$ a functor. A couniversal arrow (D, f) from C to F is uniquely determined by and uniquely determines a natural isomorphism of functors $\mathfrak{C}(C, F-) \cong \mathfrak{D}(D, -)$.*

To do Can this be phrased as an isomorphism of categories or objects? (1)

Remark (I) 2.1.9. A careful reading of the proof of prop. (I) 2.1.7 shows that, given a suitable natural isomorphism β, we were able to construct a couniversal arrow using only one, minute fragment of the information contained in β, viz., $f = \beta_D \, \mathrm{id}_D$ – the result of a single component of β on a single point. Of course, in order to give the unique arrows suitable to the definition of couniversality we need more information from β. However, in an attempt to appeal to the generosity (and *not* the philosophy) of the reader, we will claim that there is a sense in which the priority of the situation informs us that the existence of certain unique arrows $D \to D'$ scattered across $\mathfrak{D}$ precedes the data contained in β. The author bitterly regrets resorting to such appeals in this work, but it is important here that the reader engage with the forthcoming conclusion (and nothing besides).

With this in hand (or a violent disbelief thereof), the above corollary informs us that we are able to identically reconstruct β from f. This suggest that there is a sense in which β is determined by only $\beta_D \, \mathrm{id}_D$. While this may not be literally true, in section 2.2 we will find a more generally realisable result in which natural transformations *are* fully determined by microscopic facets of their data. ◁

Now, in the special case of a functor $F : \mathfrak{C} \to \textsc{Set}$ having a couniversal element we know we can uniquely associate this to a couniversal arrow and, in turn, this uniquely determines a natural isomorphism of functors involving the morphism set functor $\mathfrak{C}(C, -)$. Motivated thus, we isolate this last statement and elaborate upon the properties of functors which exhibit such an isomorphism.

Def. (I) 2.1.10. Let $\mathfrak{C}$ be locally small and $F : \mathfrak{C} \to \textsc{Set}$ be a functor. F is representable when there exists an object $C \in \mathrm{Obj}\,\mathfrak{C}$ such that $F \cong \mathfrak{C}(C, -)$ as functors. A representation of F is a pair $(C \in \mathrm{Obj}\,\mathfrak{C}, \alpha : \mathfrak{C}(C, -) \to F)$ where α is a natural isomorphism.

Prop. (I) 2.1.11. *If $\mathfrak{C}$ is locally small, a functor $F : \mathfrak{C} \to \textsc{Set}$ is representable iff it admits a couniversal element. Moreover, every representation of F uniquely determines and is uniquely determined by a couniversal element.*

Proof. The pair $(C \in \mathrm{Obj} C, x \in FC)$ is a couniversal element of F iff (prop. (I) 2.1.4) $(C, x : \{*\} \to FC)$ is a couniversal arrow from $\{*\}$ to F iff (prop. (I) 2.1.7) $\mathfrak{C}(C, -) \cong \textsc{Set}(\{*\}, F-)$ as functors. Recall that $\textsc{Set}(\{*\}, -) \cong \mathrm{id}_{\textsc{Set}}$ as functors and so composing with F we find $\mathfrak{C}(C, -) \cong \textsc{Set}(\{*\}, F-) \cong F$. The uniqueness follows from cor. (I) 2.1.8 and the proof of prop. (I) 2.1.4. ■

Remark (I) 2.1.12. **To do** (2) ◁

2.2. Yoneda embedding

We begin by giving what the author hopes appears as a small and self-evident observation to the reader. This result is largely of notational convenience.

Lem. (I) 2.2.1. *The map on objects* $h^- : \mathfrak{C}^{\mathrm{op}} \to [\mathfrak{C}, \textsc{Set}]$ *assigning* $A \mapsto h^A = \mathfrak{C}(A, -)$, *extends to a functor.*

In its most naïve form, the Yoneda lemma may be stated as follows.

Lem. (I) 2.2.2 (Yoneda). *For a small category* $\mathfrak{C}$, *any functor* $F : \mathfrak{C} \to \textsc{Set}$ *and object* $C \in \mathrm{Obj}\,\mathfrak{C}$, *there is an isomorphism of sets* $[\mathfrak{C}, \textsc{Set}](\mathfrak{C}(C,-), F) \cong FC$.

Proof. Let $\beta : FC \to [\mathfrak{C}, \textsc{Set}](\mathfrak{C}(C,-), F)$ and $\alpha : [\mathfrak{C}, \textsc{Set}](\mathfrak{C}(C,-), F) \to FC$ be defined as

$$\alpha(\tau) = \tau_C \,\mathrm{id}_C, \quad (\beta x)_A f = (Ff)x$$

for $\tau \in [\mathfrak{C}, \textsc{Set}](\mathfrak{C}(C,-), F)$, $f \in \mathfrak{C}(C, A)$, and $x \in FC$. To see that these maps are inverses of one another, consider that $\alpha\beta : FC \to FC$ and so for $x \in FC$ we have that $\alpha\beta x = \alpha(\beta x) = (\beta x)_C(\mathrm{id}_C) = (F\,\mathrm{id}_C)x = x$, ergo $\alpha\beta = \mathrm{id}_{FC}$. Then, for the reverse composition $\beta\alpha : [\mathfrak{C}, \textsc{Set}](\mathfrak{C}(C,-), F) \to [\mathfrak{C}, \textsc{Set}](\mathfrak{C}(C,-), F)$ we note that we must check that $\beta\alpha$ on every component of a natural transform τ is the identity. That is, we have $(\beta\alpha\tau)_A f = (\beta(\tau_C \,\mathrm{id}_C))_A f = Ff\tau_C \,\mathrm{id}_C$, for $\tau \in [\mathfrak{C}, \textsc{Set}](\mathfrak{C}(C,-), F)$ and $f \in \mathfrak{C}(C, A)$. In order to complete the proof we must use the naturality square of τ, from which it is apparent that $Ff\tau_C \,\mathrm{id}_C = \tau_A f$.

$$\begin{array}{ccc} \mathfrak{C}(C,C) & \xrightarrow{\tau_C} & FC \\ {\scriptstyle \mathfrak{C}(C,f)}\downarrow & & \downarrow{\scriptstyle Ff} \\ \mathfrak{C}(C,A) & \xrightarrow[\tau_A]{} & FA \end{array}$$

■

However, this is not quite the full picture. While we see now that there is a collection of isomorphisms of sets (indexed by objects and functors), what we do not yet know is how this collection of isomorphisms interacts with changes in the objects and functors. Phrased appropriately, we may recast the Yoneda lemma as the statement that the above isomorphism is natural.

Lem. (I) 2.2.3 (Yoneda). *For a small category* $\mathfrak{C}$ *write* $N, E : \textsc{Cat}(\mathfrak{C}, \textsc{Set}) \times \mathfrak{C} \rightrightarrows \textsc{Set}$ *for the functors* $N(F, C) = \textsc{Cat}(\mathfrak{C}(C,-), F)$ *and* $E(F, C) = FC$. *As functors* $N \cong E$.

As a direct consequence of the above we have the important result that the collection of natural transforms between functors of the form h^A are isomorphic to the morphisms between objects of the domain. That is,

Cor. (I) 2.2.4. $[\mathfrak{C}, \textsc{Set}](\mathfrak{C}(A,-), \mathfrak{C}(B,-)) \cong \mathfrak{C}(B, A)$.

Now we may combine this with the previous result to see that h^{-} is actually an embedding.

Cor. (I) 2.2.5 (Yoneda embedding). *The functor* $h^{-} : \mathfrak{C}^{\mathrm{op}} \to [\mathfrak{C}, \mathrm{Set}]$ *is fully faithful.*

As an immediate consequence of this fact, we are able to translate natural isomorphisms of $h^{\bullet}$ to isomorphisms of the underlying objects. In specific,

Cor. (I) 2.2.6. *If the functors* $h^{A}, h^{B} : \mathfrak{C} \rightrightarrows \mathrm{Set}$ *are naturally isomorphic then* $A \cong B$.

Proof. Fully faithful functors reflect isomorphisms. ∎

We may now reexamine a representable functors in light of this understanding. Recall that a representation of a functor F is a pair (A, τ) where $A \in \mathrm{Obj}\,\mathfrak{C}$, the representing object, and is τ a natural isomorphism to h^{A}. We see now that, due to cor. (I) 2.2.6,

Prop. (I) 2.2.7. *A representation of a functor is unique up to isomorphism.*

Due to lem. (I) 2.2.2, the natural transform τ may be uniquely identified with an element of FA – precisely our couniversal element of prop. (I) 2.1.11. If we track the proof of lem. (I) 2.2.2 carefully we can explicitly show that for $a \in FA$ a couniversal element and $b \in FB$ we determine the unique morphism $f : A \to B$ via

$$b \in FB \cong [\mathfrak{C}, \mathrm{Set}](h^{B}, F) \cong [\mathfrak{C}, \mathrm{Set}](h^{B}, h^{A}) \cong \mathfrak{C}(A, B) \ni f$$

where the first and last isomorphisms are given by the Yoneda lemma, and the middle isomorphism arises from the fact that $F \cong h^{A}$, that is, the representability of F.

Through an entirely similar series of arguments to those above, it follows that for any functor $X : \mathfrak{C}^{\mathrm{op}} \to \mathrm{Set}$ and object $C \in \mathrm{Obj}\,\mathfrak{C}$ there exists an isomorphism $[\mathfrak{C}^{\mathrm{op}}, \mathrm{Set}](\mathfrak{C}(-, C), X) \cong XC$ natural in all arguments, and so the functor $h_{-} : \mathfrak{C} \to [\mathfrak{C}^{\mathrm{op}}, \mathrm{Set}]$ is an embedding. Moreover, we are also able to carry through our arguments about representable functors to functors of the form $F : \mathfrak{C}^{\mathrm{op}} \to \mathrm{Set}$.

3. Adjunctions

Def. (I) 3.0.1. Given two categories $\mathfrak{C}$ and $\mathfrak{D}$, an adjunction between them is a pair of functors $L : \mathfrak{C} \to \mathfrak{D}$ and $R : \mathfrak{D} \to \mathfrak{C}$ such that for all objects $C \in \mathfrak{C}, D \in \mathfrak{D}$ there is an isomorphism $\mathfrak{D}(LC, D) \cong \mathfrak{C}(C, RD)$ which is natural in both arguments, that is, a natural isomorphism of morphism functors $\mathfrak{C}^{\text{op}} \times \mathfrak{D} \to \textsc{Set}$. We abbreviate this arrangement by writing $L \dashv R : \mathfrak{C} \to \mathfrak{D}$ and by stating that L is the left adjoint functor of R, or that R is the right adjoint functor of L.

Prop. (I) 3.0.2. *For functors $L \dashv R : \mathfrak{C} \to \mathfrak{D}$ the following are equivalent.*

1. $\mathfrak{D}(L-, -) \cong \mathfrak{C}(-, R-)$ *as functors* $\mathfrak{C}^{\text{op}} \times \mathfrak{D} \to \textsc{Set}$
2. *There is a natural transformation $\eta : \text{id}_{\mathfrak{C}} \to RL$ called the unit and a natural transformation $\varepsilon : LR \to \text{id}_{\mathfrak{D}}$ called the counit such that the following diagrams commute (the triangle identities)*

$$\begin{array}{ccc} L & \xrightarrow{L\eta} & LRL \\ & \diagdown\!\!\diagdown & \Big\downarrow{\scriptstyle \varepsilon L} \\ & & L \end{array} \qquad\qquad \begin{array}{ccc} R & \xrightarrow{\eta R} & RLR \\ & \diagdown\!\!\diagdown & \Big\downarrow{\scriptstyle R\varepsilon} \\ & & R \end{array}$$

3. *For every $D \in \text{Obj}\,\mathfrak{D}$ there is a universal arrow (RD, ε) from L to D where R is the resultant functor. The dual statement is equivalent, too.*
4. $(L \downarrow \text{id}_{\mathfrak{D}}) \cong (\text{id}_{\mathfrak{C}} \downarrow R)$ *and isomorphic elements in the comma categories have equal projections in $\mathfrak{C} \times \mathfrak{D}$.*

Proof (1 $\iff$ 2). To show 1 $\implies$ 2 we use a 'Yoneda style' argument. In particular, if we let $\phi_{C,D} : \mathfrak{C}(C, RD) \to \mathfrak{D}(LC, D)$ be the binatural isomorphism then we may define $\eta_C \in \mathfrak{C}(C, RLC)$ as $\eta_C = {\phi^{-1}}_{C,LC} L\,\text{id}_C$ and $\varepsilon_D \in \mathfrak{D}(LRD, D)$ as $\varepsilon_D = \phi_{RD,D} R\,\text{id}_D$. To show that these componentwise definitions do indeed yield natural transformations we must consider the binaturality of ϕ. Specifically, for $f : C \to C'$ in $\mathfrak{C}$ we have the following commutative diagram and would-be naturality square for η_C.

$$\begin{CD} \mathfrak{D}(LC, LC) @>{{\phi^{-1}}_{C,LC}}>> \mathfrak{C}(C, RLC) \\ @V{\mathfrak{D}(LC, Lf)}VV @VV{\mathfrak{C}(C, RLf)}V \\ \mathfrak{D}(LC, LC') @>{{\phi^{-1}}_{C,LC'}}>> \mathfrak{C}(C, RLC') \\ @A{\mathfrak{D}(Lf, LC')}AA @AA{\mathfrak{C}(f, RLC')}A \\ \mathfrak{D}(LC', LC') @>{{\phi^{-1}}_{C',LC'}}>> \mathfrak{C}(C', RLC') \end{CD} \qquad\qquad \begin{CD} C @>{\eta_C}>> RLC \\ @V{f}VV @VV{RLf}V \\ C' @>>{\eta_{C'}}> RLC' \end{CD}$$

We wish to show that the right diagram commutes and to do so we start with id_{LC} and $\mathrm{id}_{LC'}$ in the left corners of the left diagram to find that

$$RLf\eta_C = RLf\phi^{-1}{}_{C,LC}\,\mathrm{id}_{LC} = \phi^{-1}{}_{C,LC'}Lf = \phi^{-1}{}_{C',LC'}f = \eta_{C'}f$$

The case for ε is analogous, but makes use of the naturality of ϕ instead. Finally, we verify that the triangle identities hold.

Observe that the following diagrams commute for $f' : C \to RD$ and $g' : LC \to D$ by the naturality of $\phi_{\bullet,D}$ and $\phi^{-1}{}_{C,\bullet}$ respectively.

$$\begin{array}{ccc} \mathfrak{C}(RD,RD) & \xrightarrow{\phi_{RD,D}} & \mathfrak{D}(LRD,D) \\ {\scriptstyle \mathfrak{C}(f',RD)}\downarrow & & \downarrow{\scriptstyle \mathfrak{D}(Lf',D)} \\ \mathfrak{C}(C,RD) & \xrightarrow[\phi_{C,D}]{} & \mathfrak{D}(LC,D) \end{array} \qquad \begin{array}{ccc} \mathfrak{D}(LC,LC) & \xrightarrow{\phi^{-1}{}_{C,LC}} & \mathfrak{C}(C,LRC) \\ {\scriptstyle \mathfrak{D}(LC,g')}\downarrow & & \downarrow{\scriptstyle \mathfrak{C}(C,Rg')} \\ \mathfrak{D}(LC,D) & \xrightarrow[\phi^{-1}{}_{C,D}]{} & \mathfrak{C}(C,RD) \end{array}$$

Tracing out these diagrams for id_{RD} and id_{LC}, and making use of the fact the ϕ is an isomorphism to rewrite the arrows $f' = \phi^{-1}{}_{C,D}f$ and $g' = \phi_{C,D}g$, we find the equations $(\phi_{RD,D}\,\mathrm{id}_{RD})L(\phi^{-1}{}_{C,D}f) = f$ and $R(\phi_{C,D}g)(\phi^{-1}{}_{C,LC}\,\mathrm{id}_{LC}) = g$. Consequently, for arbitrary $C \in \mathrm{Obj}\,\mathfrak{C}$ and $D \in \mathrm{Obj}\,\mathfrak{D}$ where we specialise the statements of the triangle identities we have

$$\varepsilon_{LC}L\eta_C = (\phi_{RLC,LC}\,\mathrm{id}_{RLC})L(\phi^{-1}_{C,LC}\,\mathrm{id}_{LC}) = \mathrm{id}_{LC}$$

$$R\varepsilon_D\eta_{RD} = R(\phi_{RD,D}\,\mathrm{id}_{RD})(\phi^{-1}{}_{RD,LRD}\,\mathrm{id}_{LRD}) = \mathrm{id}_{RD}$$

thereby completing the proof of $1 \implies 2$.

To show that $2 \implies 1$ we begin by defining $\phi : \mathfrak{C}(\bullet, R\bullet) \to \mathfrak{D}(L\bullet,\bullet)$ and its to-be inverse on their components. For $C \in \mathrm{Obj}\,\mathfrak{C}$ and $D \in \mathrm{Obj}\,\mathfrak{D}$, let $\phi_{C,D} = \varepsilon_D L$ and $\phi^{-1}{}_{C,D}f = Rf\eta_C$ for $f \in \mathfrak{D}(LC,D)$. We first check that these maps are indeed inverses and then proceed to show binaturality. Observe that for all $f : C \to RD$

$$\begin{aligned} \phi^{-1}{}_{C,D}\phi_{C,D}f &= R\varepsilon_D RLf\eta_C && \text{(definition)} \\ &= R\varepsilon_D\eta_{RD}f && \text{(naturality of } \eta) \\ &= (\mathrm{id}_R)_D f && \text{(triangular identity)} \end{aligned}$$

Similarly, we show that $\phi_{C,D}\phi^{-1}{}_{C,D} = \mathrm{id}_{\mathfrak{D}(LC,D)}$ by making use of the appropriate naturality statement for ε and the other triangular identity. With the inverse property established, we must demonstrate the binaturality of $\varphi_{C,D}$ and its inverse.

In the case of the former, for fixed $D \in \mathrm{Obj}\,\mathfrak{D}$ and $f : D \to D'$ we wish to show that $\mathfrak{D}(LC,f)\phi_{C,D}g = f\varepsilon_D Lg$ and $\phi_{C,D'}\mathfrak{C}(C,Rf)g = \varepsilon_{D'}LRfLg$ are equal for all $g \in \mathfrak{C}(C,RD)$. However, $\varepsilon_{D'}LRf = f\varepsilon_D$ by naturality and so $\phi_{C,-}$ is natural.

In the case of $\phi_{-,D}$, we consider $C \in \mathrm{Obj}\,\mathfrak{C}$ and $f : C' \to C$ in $\mathfrak{C}^{\mathrm{op}}$ and desire that $\mathfrak{D}(Lf,D)\phi_{C',D}g = \varepsilon_D LgLf$ equals $\phi_{C,D}\mathfrak{C}(f,RD)g = \varepsilon_D L(gf)$ for all $g \in \mathfrak{C}(C',RD)$. Here, the proof is truly trivial. ■

Proof (2 $\iff$ 3). The essence of the proof was given in the proof of prop. (I) 1.0.7, but the relevant details are repeated here.

For a given $D \in \mathrm{Obj}D$ we can form (RD, ε_D) as an object of $(L \downarrow D)$. To show that it is terminal requires showing that for each $(C, f : LC \to D)$ in $(L \downarrow D)$ we have a unique arrow $u : C \to RD$ such that $\varepsilon_D Lu = f$. Inspired by earlier work (what amounts to the dual proof in prop. (I) 1.0.7), we set $u = Rf\eta_C$ and check that $\varepsilon_D LRfL\eta_C = f\varepsilon_{LC}L\eta_C = f$ by naturality and the first triangle identity. Then, for uniqueness we assume that there exists a $v : C \to RD$ which satisfies $\varepsilon_D Lv = f$. If it does, we may simply expand $R(\varepsilon_D Lv)\eta_C = Rf\eta_C = u$ but also $R\varepsilon_D RLv\eta_C = R\varepsilon_D\eta_{RD}v = v$ by naturality and the other triangle identity, and so $v = u$. The other direction is the statement of prop. (I) 1.0.7. What remains follows easily by dualisation. ■

Prop. (I) 3.0.3. *Adjoints are unique up to isomorhpism.*

Proof. Suppose that $L \dashv R, R' : \mathfrak{C} \to \mathfrak{D}$, then in particular we have natural isomorphisms $\alpha : \mathfrak{C}(-, R-) \to \mathfrak{D}(L-, -)$ and $\beta : \mathfrak{D}(L-, -) \to \mathfrak{C}(-, R'-)$ and so a natural isomorphism $\beta\alpha : \mathfrak{C}(-, R-) \to \mathfrak{C}(-, R')$. Consequently, for every $D \in \mathrm{Obj}\,\mathfrak{D}$, $h_{RD} \cong h_{RD'}$ and so by Yoneda, $RD \cong R'D$. Let $\gamma_D : RD \to R'D$ be the ismorophism with $h^C\gamma_D = (\beta\alpha)_{C,D}$, and let $f : D \to E$ be an arrow in $\mathfrak{D}$. Consider that the diagram below left commutes iff the diagram below right commutes, as h^C is fully faithful for every $C \in \mathrm{Obj}C$, by Yoneda.

$$
\begin{array}{ccc}
h^C RD & \xrightarrow{(\beta\alpha)_{C,D}} & h^C R'D \\
{\scriptstyle h^C Rf}\downarrow & & \downarrow{\scriptstyle h^C R'f} \\
h^C RE & \xrightarrow[(\beta\alpha)_{C,E}]{} & h^C R'E
\end{array}
\qquad\qquad
\begin{array}{ccc}
RD & \xrightarrow{\gamma_D} & R'D \\
{\scriptstyle Rf}\downarrow & & \downarrow{\scriptstyle R'f} \\
RE & \xrightarrow[\gamma_E]{} & R'E
\end{array}
$$

However, the diagram on the left commutes by the naturality of $(\beta\alpha)$ and so $R \cong R'$ via γ, naturally. That left adjoints are unique follows by dualisation. ■

Prop. (I) 3.0.4. *If* $L \dashv R : \mathfrak{B} \to \mathfrak{C}$ *and* $L' \dashv R' : \mathfrak{C} \to \mathfrak{D}$ *then* $L'L \dashv RR' : \mathfrak{B} \to \mathfrak{D}$.

Proof. $\mathfrak{D}(L'LB, D) \cong \mathfrak{C}(LB, R'D) \cong \mathfrak{B}(B, RR'D)$. ■

Prop. (I) 3.0.5. *Any functor* $R : \mathfrak{C} \to \mathrm{SET}$ *with left adjoint* $L : \mathrm{SET} \to \mathfrak{C}$ *is representable, and has couniversal object and element* $(\{\star\}, \eta_{\{\star\}}(\star))$.

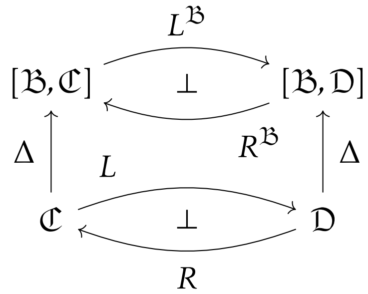
Proof. If we show that $R \cong \mathrm{SET}(\{\star\}, R-)$, then it follows by adjunction that we have $R \cong \mathrm{SET}(\{\star\}, R-) \cong \mathfrak{C}(L\{\star\}, -)$ and so R is representable.

To that end, define $\sigma : R \to \mathrm{SET}(\{\star\}, R-)$ as $\sigma_C(c)(\star) = c$ and define ${\sigma^{-1}}_C(f) = f\star$. It is clear that σ_C and ${\sigma^{-1}}_C$ are inverses, so we must show that they are natural. Fortunately, this is trivial as we merely desire that for $f : C \to D$ $Rf\sigma_C = \sigma_D Rf$ but we have $(Rf(\sigma_C(c)))(\star) = Rfc = (\sigma_D Rf(c))(\star)$ and similarly for σ^{-1}. Thus σ is a natural isomorphism and so, by prop. (I) 2.1.11 and remark (I) 2.1.12, we have that $(\{\star\}, \eta_{\{\star\}}\mathrm{id}_{\{\star\}})$ is a couniversal element of R. ■

Prop. (I) 3.0.6. *If* $\mathfrak{C}$ *has all limits of shape* $\mathfrak{B}$, *then* $\Delta \dashv \lim : \mathfrak{C} \to [\mathfrak{B}, \mathfrak{C}]$.

Proof. We defined limits in terms of universal arrows in section 1 and so the statement is an immediate consequence of prop. (I) 3.0.2 (3). ∎

Lem. (I) 3.0.7. *If $L \dashv R : \mathfrak{C} \to \mathfrak{D}$ then $L^{\mathfrak{B}} \dashv R^{\mathfrak{B}} : [\mathfrak{B}, \mathfrak{C}] \to [\mathfrak{B}, \mathfrak{D}]$ and the following diagram commutes, for any category $\mathfrak{B}$.*



Proof. Let $L \dashv R$ have unit and counit η, ϵ and define $\eta^{\mathfrak{B}} : \mathrm{id}_{[\mathfrak{B},\mathfrak{C}]} \to R^{\mathfrak{B}} L^{\mathfrak{B}}$ through components as ${\eta^{\mathfrak{B}}}_F = \eta F$, and similarly ${\varepsilon^{\mathfrak{B}}}_G = \varepsilon G$ for $F \in [\mathfrak{B}, \mathfrak{C}]$ and $G \in [\mathfrak{B}, \mathfrak{D}]$. We show that $\eta^{\mathfrak{B}}, \varepsilon^{\mathfrak{B}}$ are natural and that the triangle identities are obeyed.

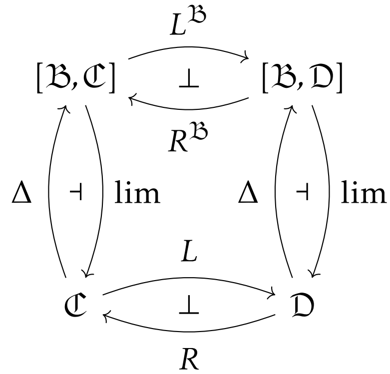
To see that $\eta^{\mathfrak{B}}$ is natural, let $F, F' \in [\mathfrak{B}, \mathfrak{C}]$ and $\tau : F \to F'$ natural between them. We wish to have ${\eta^{\mathfrak{B}}}_F R^{\mathfrak{B}} L^{\mathfrak{B}} \tau = \tau {\eta^{\mathfrak{B}}}_{F'}$. However, the left hand side is just $(\eta F)(RL\tau) = \tau \eta_{F'}$ by naturality of η. The proof for $\varepsilon^{\mathfrak{B}}$ is entirely similar.

That the triangle identities hold is equally trivial, as we are merely dealing with η and ε with components as the image of a functor. Specifically, we may expand $(\varepsilon^{\mathfrak{B}} L^{\mathfrak{B}})_F (L^{\mathfrak{B}} \eta^{\mathfrak{B}})_F = (\varepsilon L F)(L \eta F) = (\varepsilon L)(L \varepsilon) F = F$.

Finally, that the diagram commutes is also something of a triviality, in that we wish to show that $L^{\mathfrak{B}} \Delta = \Delta L$ and $R^{\mathfrak{B}} \Delta = \Delta R$. Of course, $L^{\mathfrak{B}} \Delta C = L \Delta C = \Delta L C$ for every $C \in \mathrm{Obj}\, \mathfrak{C}$, thereby concluding the proof. ∎

Prop. (I) 3.0.8. *Right adjoints are continuous.*

Proof. Let $L \dashv R : \mathfrak{C} \to \mathfrak{D}$ and $\mathfrak{B}$ be a small category with $\mathfrak{C}$ and $\mathfrak{D}$ having all limits of shape $\mathfrak{B}$, and consider the following diagram.



In order to show the continuity of R, we must show that if $\alpha : \Delta \lim F \to F$ is the limiting cone for F then it must be that $R\alpha : R \Delta \lim F \to RF$ is the limiting cone for RF. Thus, let $\beta : \Delta C \to RF$ be a cone over RF and ${\Phi^{-1}}_{\Delta C, F} : [\mathfrak{B}, \mathfrak{C}](\Delta C, R^{\mathfrak{B}} F) \to$

$[\mathfrak{B},\mathfrak{D}](L^{\mathfrak{B}}\Delta C,F)$ be the binatural isomorphism arising from $L^{\mathfrak{B}} \dashv R^{\mathfrak{B}}$. As Φ^{-1} is an isomorphism, it is clear that ${\Phi^{-1}}_{\Delta C,F}\beta : \Delta LC \to F$ is a cone for F. As such, there exists a unique $u : LC \to \lim F$ such that ${\Phi^{-1}}_{\Delta C,F}\beta = \alpha\Delta u$.

If we write $\phi_{C,\lim F} : \mathfrak{D}(LC, \lim F) \to \mathfrak{C}(C, R\lim F)$ as the binatural isomorphism arising from $L \dashv R$, then we may note that $\phi_{C,\lim F}u : C \to R\lim F$ and further that, for $B \in \operatorname{Obj}\mathfrak{B}$

$$
\begin{aligned}
(R\alpha)_B\Delta(\phi_{C,\lim F}u) &= R\alpha_B\phi_{C,\lim F}u \\
&= \phi_{C,FB}(\alpha_B u) && \text{(naturality of } \phi_{C,-}\text{)} \\
&= \phi_{C,FB}\,({\Phi^{-1}}_{\Delta C,F}\beta)_B && ({\Phi^{-1}}_{\Delta C,F}\beta = \alpha\Delta u) \\
&= \phi_{C,FB}{\phi^{-1}}_{C,FB}\beta_B && (*) \\
&= \beta_B
\end{aligned}
$$

To see the equality marked $(*)$ we recall that ${\Phi^{-1}}_{\Delta C,F} = \varepsilon^{\mathfrak{B}}L^{\mathfrak{B}}$. With this in hand, to show that $R\alpha$ is a limiting cone has been reduced to the task of showing that $\phi_{C,FB}u$ is unique.

Suppose there was an arrow $v : C \to R\lim F$ such that $R\alpha\Delta v = \beta$, we may check that $\alpha_B{\phi^{-1}}_{C,\lim F}v = {\phi^{-1}}_{C,FB}\alpha_B v = \phi_{C,FB}\beta_B$ and so $\alpha\Delta{\phi^{-1}}_{C,\lim F}v = \Phi_{\Delta C,F}\beta$, forcing the equality ${\phi^{-1}}_{C,\lim F}v = u$ by universality of u for cones over F. Ergo, $R\alpha$ is a limiting cone for $R\lim F$.

Finally, we know that we may compose adjoints to find $L^{\mathfrak{B}}\Delta \dashv \lim R^{\mathfrak{B}}$, but we have $L^{\mathfrak{B}}\Delta = \Delta L$ and $\Delta L \dashv R\lim$ and so by prop. (I) 3.0.3 it must be the case that $\lim R^{\mathfrak{B}} \cong R\lim$ as functors $[\mathfrak{B},\mathfrak{C}] \to \mathfrak{D}$. Thus, for any given functor $F : \mathfrak{B} \to \mathfrak{D}$, $\lim RF \cong R\lim F$. ■

3.1. Theorems

To do Adjoint functor theorems (3) **To do** Reflective subcategories? (4)

3.2. Free objects

To do Free-forgetful adjunction (5)

3.3. Miscellaneous examples

To do Anything that will prove useful later (6)

4. Monads

4.1. Algebras

To do Kleisli, Eilenburg-Moore? (7)

II. Monoidal categories

1. Basic notions

A simple starting point in endowing a category with extra structure would be to somehow induce an algebraic structure upon its objects. Of the various algebraic structures at hand, monoids stand out as demanding relatively little in the way of structure, but offering a rich enough theory to be of interest. As such, we will attempt to view $\mathrm{Obj}\,\mathfrak{C}$ as a monoid.

Def. (II) 1.0.1. A monoidal category $\mathfrak{C}$ is a category equipped with

1. a bifunctor $\otimes : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$, the tensor product
2. an object $I \in \mathrm{Obj}\,\mathfrak{C}$, the identity object
3. a natural isomorphism $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, the associator
4. a natural isomorphism $\lambda : I \otimes \bullet \to \mathrm{id}_{\mathfrak{C}}$, the left unitor
5. a natural isomorphism $\rho : \bullet \otimes I \to \mathrm{id}_{\mathfrak{C}}$, the right unitor

such that the following diagrams commute for all $A, B, C, D \in \mathrm{Obj}\,\mathfrak{C}$. The monoidal category is called strict when α, λ, ρ are identities.

$$\begin{array}{ccc} (A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} & A \otimes (I \otimes B) \\ {\scriptstyle \rho_A \otimes \mathrm{id}_B} \searrow & & \swarrow {\scriptstyle \mathrm{id}_A \otimes \lambda_B} \\ & A \otimes B & \end{array}$$

$$\begin{array}{ccccc} & & ((A \otimes B) \otimes C) \otimes D & & \\ & {\scriptstyle \alpha_{A,B,C} \otimes \mathrm{id}_D} \swarrow & & \searrow {\scriptstyle \alpha_{A \otimes B,C,D}} & \\ (A \otimes (B \otimes C)) \otimes D & & & & (A \otimes B) \otimes (C \otimes D) \\ \downarrow {\scriptstyle \alpha_{A,B \otimes C,D}} & & & & \downarrow {\scriptstyle \alpha_{A,B,C \otimes D}} \\ A \otimes ((B \otimes C) \otimes D) & & \xrightarrow[\mathrm{id}_A \otimes \alpha_{B,C,D}]{} & & A \otimes (B \otimes (C \otimes D)) \end{array}$$

Note that we do not require the components of the unitors and associator to be identities, only isomorphisms. Moreover, naturality ensures that they commute with arrows in the category – if $f : A \to B$ then $\lambda_B(\mathrm{id}_I \otimes f) = f\lambda_A$ as arrows from $I \otimes A$ to B, for instance – thereby maximally preserving the monoidal structure.

If we truly wish to express the difference between the monoidal category $(\mathfrak{C}, \otimes, I, \alpha, \lambda, \rho)$ and the 'underlying' category $\mathfrak{C}$, we will write $\mathfrak{C}_0$ for the latter. In this way, a functor $F : \mathfrak{C}_0 \to \mathfrak{D}_0$ is an 'ordinary' functor.

Another notational convention is the omission of the $\otimes$ symbol between objects in favour of juxtaposition where unambiguous, and the writing of $\mathrm{id}_A \otimes f$ and $f \otimes \mathrm{id}_A$ as Af and fA respectively. This convention has the convenient side effect that some identities are obvious ($\mathrm{id}_A \otimes \mathrm{id}_B = \mathrm{id}_{A \otimes B}$ just reads $AB = AB$).

Before we divert the reader's attention with examples and surrounding theory, there is an immediate question which may seem, at first glance, somewhat troublesome. Notice that both λ_I and ρ_I are natural isomorphisms $II \to I$. We do not explicitly require these to coincide, and it may seem strange to leave such matters up to chance. To quell such unsettling ideas, we give the following results.

Prop. (II) 1.0.2. *In any monoidal category, the following hold for all objects A and B:*

1. $\lambda_{IA} = I\lambda_A$ *and* $\rho_{AI} = \rho_A I$
2. $\lambda_{AB}\alpha_{I,A,B} = \lambda_A B$ *and* $\rho_{AB} = (A\rho_B)\alpha_{A,B,I}$
3. $\lambda_I = \rho_I$

Proof. The first statement follows from the naturality squares for λ and ρ. Observe that $\lambda_A \lambda_{IA} = \lambda_A(I\lambda_A)$ and similarly for ρ, and as λ, ρ are isomorphisms we have (1).

With this established, we turn to the proof of (2). Though perhaps long, it is entirely mechanical relying only on the naturality of α and the pentagonal and triangular identities in def. (II) 1.0.1. For these reasons, we explicitly only show the first statement, as the second is entirely similar.

To begin, observe that if we wish to show $f = g$ for generic arrows $f, g : A \rightrightarrows B$, it suffices to show that $If = Ig$ as then $\lambda_B(If) = \lambda_B(Ig)$ but by naturality $\lambda_B(If) = f\lambda_A$ and $f\lambda_A = g\lambda_A \implies f = g$, that is, λ and ρ give an equivalence of categories where the functors are $I\otimes$ and $\otimes I$ respectively.

With that established, we will show that $I(\lambda_{AB}\alpha_{I,A,B}) = I(\lambda_A B)$.

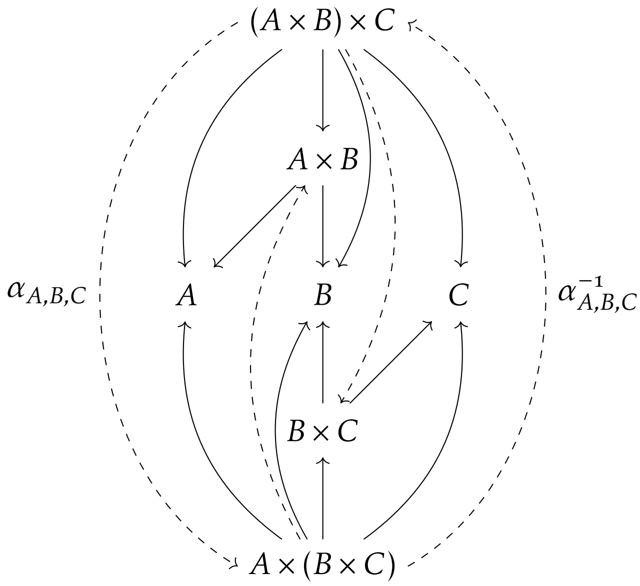
$$
\begin{aligned}
& I(\lambda_{AB}\alpha_{I,A,B}) = (I\lambda_{AB})(I\alpha_{I,A,B}) \\
&= (I\lambda_{AB})\alpha_{I,I,AB}{\alpha^{-1}}_{I,I,AB}(I\alpha_{I,A,B}) && \text{(triangular identity)} \\
&= (\rho_I(AB)){\alpha^{-1}}_{I,I,AB}(I\alpha_{I,A,B}) \\
&= \alpha_{I,A,B}{\alpha^{-1}}_{I,A,B}(\rho_I(AB)){\alpha^{-1}}_{I,I,AB}(I\alpha_{I,A,B}) \\
&= \alpha_{I,A,B}((\rho_I A)B){\alpha^{-1}}_{II,A,B}{\alpha^{-1}}_{I,I,AB}(I\alpha_{I,A,B}) && \text{(naturality of } \alpha^{-1}) \\
&= \alpha_{I,A,B}((I\lambda_A)\alpha_{I,I,A}B){\alpha^{-1}}_{II,A,B}{\alpha^{-1}}_{I,I,AB}(I\alpha_{I,A,B}) && \text{(triangular identity)} \\
&= \alpha_{I,A,B}((I\lambda_A)B)(\alpha_{I,I,A}B){\alpha^{-1}}_{II,A,B}{\alpha^{-1}}_{I,I,AB}(I\alpha_{I,A,B}) \\
&= (I(\lambda_A B))\alpha_{I,IA,B}(\alpha_{I,I,A}B){\alpha^{-1}}_{II,A,B}{\alpha^{-1}}_{I,I,AB}(I\alpha_{I,A,B}) && \text{(naturality of } \alpha) \\
&= I(\lambda_A B) && \text{(pentagonal identity)}
\end{aligned}
$$

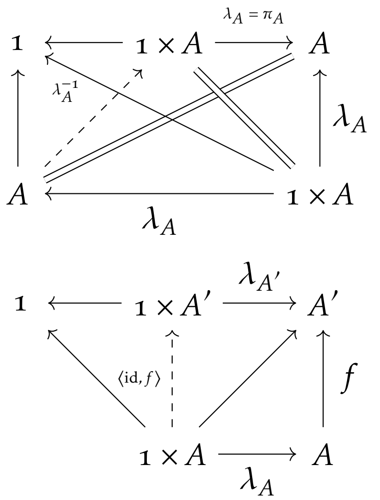
Finally, we note that the equality $\lambda_I I \overset{(2)}{=} \lambda_{II}\alpha_{I,I,I} \overset{(1)}{=} (I\lambda_I)\alpha_{I,I,I} \overset{\text{(tri.)}}{=} \rho_I I$ is sufficient to give (3), and thereby conclude the proof. ■

With that established, we divert our attention to examples of monoidal categories.

Example (II) 1.0.3

A cartesian monoidal category is a category $\mathfrak{C}$ which supports all finite products, endowed with the monoidal structure of $\otimes = \times$, $I = 1$ and $\alpha_{A,B,C}, \lambda_A, \rho_A$ the canonical isomorphisms arising from the below commutative diagrams.





That ρ, ρ^{-1} are natural isomorphisms is similarly straightforward to see. That α is natural is also true, but not demonstrated explicitly in the above diagrams.

The reader should be quick to note that there is a dual to the above, a *cocartesian* monoidal category. If a category supports all finite *coproducts* then we may view $(\mathfrak{C}, 0, +)$ as a monoidal category with α, λ, ρ the canonical isomorphisms.

As in the above example, there are many situations in which the tensor product behaves like a product in some sense of the word. Perhaps an important class of such examples is when the tensor product is actually a tensor product of modules or vector spaces.

Example (II) 1.0.4

The category Ab of abelian groups admits a monoidal structure. In particular, if we view abelian groups as $\mathbb{Z}$-modules then the bifunctor can be the tensor product of $\mathbb{Z}$-modules, $\otimes_{\mathbb{Z}}$, with $\mathbb{Z}$ serving as the identity and α, λ, ρ the canonical isomorphisms. That is, we would define $\lambda_G(n \otimes g) = ng$ and $\lambda^{-1}{}_G(ng) = 1 \otimes ng$, and similarly for ρ, from which it is easy to check that all the requisite diagrams commute.

Finally, to demonstrate that tensor products needn't be products in any sense of the word, we move to examine the category of endofunctors on a category.

Example (II) 1.0.5

> The category $[\mathfrak{C},\mathfrak{C}] = \operatorname{End}\mathfrak{C}$ is monoidal with the tensor product being composition of functors, the Godement product on natural transformations, and the identity object as $\mathrm{id}_{\mathfrak{C}}$. Furthermore, $\alpha_{A,B,C} = \mathrm{id}_{ABC}$ as composition is associative, and it is readily apparent that $\lambda_A = \rho_A = \mathrm{id}_A$ and that α, β, ρ are natural. In fact, the category of endofunctors is thus a strict monoidal category under functor composition.

Prop. (II) 1.0.6. *The unit object in a monoidal category is unique up to isomorphism.*

Proof. Let $\mathfrak{C}$ be a monoidal category with unit objects I and I', and associated unitors λ, ρ and λ', ρ'. Observe that $f = \lambda_{I'}(\rho'_I)^{-1} : I \to I'$ is an isomorphism as it is the composite of two isomorphisms. **To do** Show this isomorphism is unique (8) ■

Given this fact, and drawing on inspiration from Set and Cat, we will begin to think of arrows from I to an object of the category C as *generalised elements* of C. Although the morphisms are not in any meaningful way 'contained in' C, they do (in many ways) speak for the way that C 'behaves'. The reader is advised to entertain this notion and be aware of how it repeatedly reappears in the context of monoidal categories (c.f. monoid objects, closed monoidal categories, *etc*).

Now that we have a working understanding of how monoidal categories may manifest themselves, and which abstractions they attempt to capture, we may ask the natural question: do they form a category? In order to answer this, we must first define functors between monoidal categories. In doing so, we notice that there are varying degrees to which we may seek to have the functor respect the monoidal nature of the categories at hand.

Def. (II) 1.0.7. Given two monoidal categories $(\mathfrak{C}, I_{\mathfrak{C}})$ and $(\mathfrak{D}, I_{\mathfrak{D}})$, a monoidal functor between them is given by a triplet (F, ϕ, ε) where $F : \mathfrak{C}_0 \to \mathfrak{D}_0$ is a functor, $\phi_{A,B} : FAFB \to F(AB)$ is a natural transformation and $\varepsilon : I_{\mathfrak{D}} \to FI_{\mathfrak{C}}$ is a morphism in $\mathfrak{D}$, such that for all $A, B, C \in \operatorname{Obj} C$ the following diagrams commute.

$$\begin{array}{ccc}
(FAFB)FC & \xrightarrow{\alpha_{\mathfrak{D}}} & FA(FBFC) \\
{\scriptstyle \phi_{A,B}FC}\downarrow & & \downarrow{\scriptstyle FA\phi_{B,C}} \\
F(AB)FC & & FAF(BC) \\
{\scriptstyle \phi_{AB,C}}\downarrow & & \downarrow{\scriptstyle \phi_{A,BC}} \\
F((AB)C) & \xrightarrow[F\alpha_{\mathfrak{C}}]{} & F(A(BC))
\end{array}$$

$$\begin{array}{ccc}
I_{\mathfrak{D}}FA & \xrightarrow{\lambda_{\mathfrak{D}}} & FA \\
{\scriptstyle \varepsilon FA}\downarrow & & \uparrow{\scriptstyle F\lambda_{\mathfrak{C}}} \\
FI_{\mathfrak{C}}FA & \xrightarrow[\phi_{I_{\mathfrak{C}},A}]{} & F(I_{\mathfrak{C}}A)
\end{array}
\qquad
\begin{array}{ccc}
FAI_{\mathfrak{D}} & \xrightarrow{\rho_{\mathfrak{D}}} & FA \\
{\scriptstyle FA\varepsilon}\downarrow & & \uparrow{\scriptstyle F\rho_{\mathfrak{C}}} \\
FAFI_{\mathfrak{C}} & \xrightarrow[\phi_{A,I_{\mathfrak{C}}}]{} & F(AI_{\mathfrak{C}})
\end{array}$$

The first diagram expresses how ϕ, the 'factoring' operation, and F should respect the associative nature of the monoidal categories. If both categories are strict, then the diagram reduces to a commuting square expressing simply that the order in which we 'factor' out F does not matter. The next two diagrams express for us that 'factoring' out F should respect the left and right unitors of the categories. This, the reader should bear in mind, is analogous to the idea of monoid homomorphisms for which we have $f(a)f(b)f(c) = f(abc)$, among other identities.

The reader would do well to scrutinise the previous statement. Indeed, the above diagram essentially only gives 'one direction' of the identity present for monoid homomorphisms, viz., $f(a)f(b)f(c) \to f(abc)$. In particular, no restrictions were placed on the invertibility of the 'factoring' operation or indeed the unit morphism. It is thus here that we have an opportunity to discriminate among monoidal functors, through the degree to which they are true to our analogy.

Def. (II) 1.0.8. A monoidal functor (F, ϕ, ε) is termed

- Lax, if it merely satisfies the conditions present in def. (II) 1.0.7.
- Strong, if ϕ is a natural isomorphism and ε is an isomorphism.
- Strict, if ϕ and ε are identities.

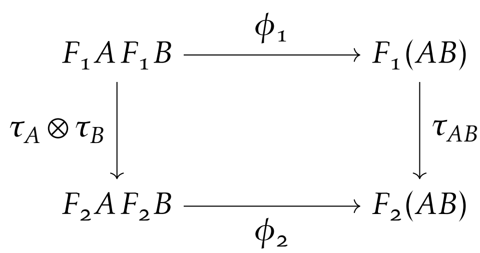
Prop. (II) 1.0.9. *With the following composition of monoidal functors and the evident identity monoidal functor,* MonCat *is endowed with a category structure.*

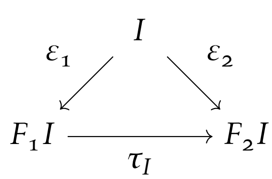
$$(F', \phi', \varepsilon') \circ (F, \phi, \varepsilon) = (F'F, (F'\phi)(\phi'_{F\times F}), (F'\varepsilon)\varepsilon')$$

Moreover, the composite of two monoidal functors yields a third of strength equal to the minimum of the two.

As will prove useful later, with a basic understanding of monoidal functors between monoidal categories in hand, we move to briefly address the notion of monoidal natural transforms between monoidal functors. Intuitively, we expect that a monoidal natural transform should respect the functorial nature in the ordinary way, that it should respect 'factoring' morphism in a natural manner, and that it should take identity morphisms to identity morphisms. Indeed,

Def. (II) 1.0.10. If $F_1, F_2 : \mathfrak{C} \rightrightarrows \mathfrak{D}$ are two monoidal functors between monoidal categories then a monoidal natural transform $\tau : F_1 \to F_2$ is a natural transform of the functors $F_1, F_2 : \mathfrak{C}_0 \rightrightarrows \mathfrak{D}_0$ such that the following diagrams commute for all $A, B \in \operatorname{Obj} C$.





1.1. Braiding and symmetry

Now that we have a grasp on the elementary notions concerning monoidal categories, we may be tempted to investigate certain properties of such categories. In particular, should we examine Set (or indeed any cartesian monoidal category) we notice that we have $A \times B \cong B \times A$ naturally in both arguments. However, it is clear that this is not the case in a general monoidal category. Despite being strict, End $\mathfrak{C}$ does not exhibit such behaviour and thus we must introduce this at the level of a structural property.

However, there is another hidden privilege that cartesian monoidal categories enjoy. If we let $\beta_{A,B} : A \times B \to B \times A$ be the natural isomorphism then it is a matter of some triviality that $\beta_{A,B}\beta_{B,A} = \mathrm{id}_{A \times B}$. However, as the example below illustrates, this is not generally the case.

Example (II) 1.1.1

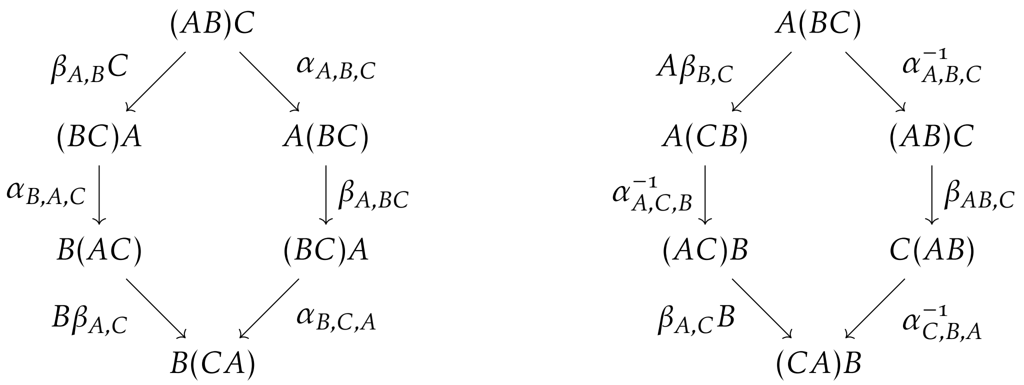
Recall that a ring R is N-graded, for a monoid N, if there exist a family of subgroups $(R_n)_{n \in N}$ such that $R = \bigoplus R_n$ and $R_n \cdot R_m \subseteq R_{n+m}$. Further, recall that if M is a right R-module, M is a graded right R-module if there exist a family of subgroups $(M_n)_{n \in N}$ such that $M = \bigoplus M_n$ and $M_m \cdot R_n \subseteq M_{m+n}$.

If R is a graded commutative ring, then we may form GMod_R, the category of graded R-modules, whose objects are graded R-modules and whose morphisms are graded module morphisms. That is, if $f : M \to N$ for $M, N \in \mathrm{Obj}\, \mathrm{GMod}_R$ then $f(M_n) \subseteq N_n$ and f is otherwise an R-module morphism.

It is a simple matter to verify that defining $(M \otimes N)_k = \sum_{m+n=k} M_m \otimes_R N_n$ endows the category with a monoidal structure. Moreover, it can be shown [JS68] that braidings for GMod_R are in bijection with invertible elements r of R, and are given by $\beta_{M,N}(a \otimes b) = r^{mn}(b \otimes a)$ for $a \in M_m$ and $b \in N_n$. It is clear that, in general, $\beta^2 \neq \mathrm{id}$.

Given this example, it is clear now that we cannot in general require that $\beta^2 = \mathrm{id}$ as it is in the cartesian monoidal case. However, the general notion stands and we define, with suitable coherence requirements,

Def. (II) 1.1.2. A braided monoidal category is a monoidal category equipped with a binatural isomorphism $\beta_{A,B} : AB \to BA$ such that the following diagrams commute for all $A, B, C \in \mathrm{Obj}\, \mathfrak{C}$.

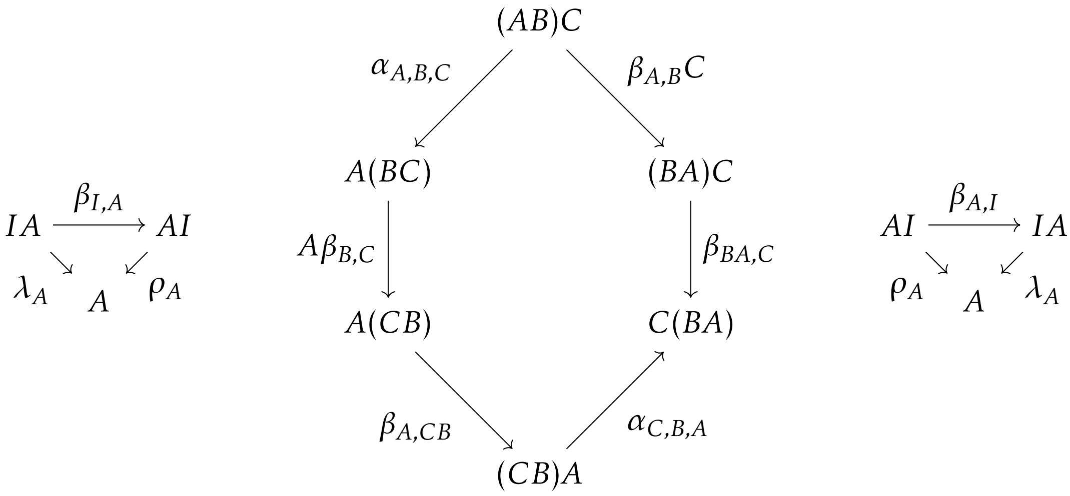


In the above, the second diagram can be seen to be identical to the first with $\beta_{A,B}$ replaced by ${\beta^{-1}}_{B,A}$, thereby indicating that β and β^{-1} are allowed to be somehow 'different'. Moreover, the first diagram encapsulates the notion that we may either braid A

through BC in 'one-step' or successively 'pull' it through B and then C, and then reach the same result. Similarly, the second diagram says that we may braid AB through C in 'one-step' or gradually, without changing the result. In a strict monoidal category, the diagrams give $\beta_{A,BC} = (B\beta_{A,C})(\beta_{A,B}C)$ and $\beta_{AB,C} = (\beta_{A,C}B)(A\beta_{B,C})$.

As we may expect, these coherence conditions suffice to prove that the braiding respects 'reasonable' operations (in particular, unitors and associators). The following is a theorem of [JS68], given here without proof.

Thm. (II) 1.1.3 (Joyal, Street). *In a braided monoidal category, the following diagrams commute.*

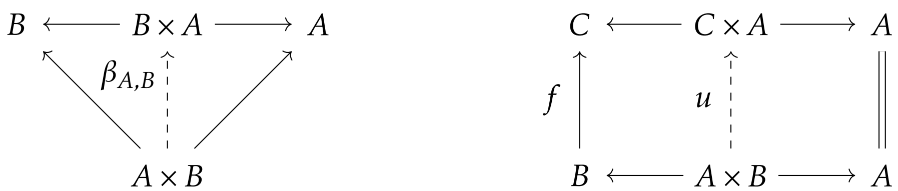


Of course, to justify our generalisation we are able to show that cartesian monoidal categories are canonically braided.

Prop. (II) 1.1.4. *Every cartesian monoidal category admits a canonical braiding.*

Proof. First we define β, then we show it is natural, and finally demonstrate that the left hexagonal diagram commutes – the proof of the right one is entirely similar.

We define β through the universal property in the below-left diagram, and then make use of the below-right diagram to show its naturality in its second argument – the case for the first is entirely similar. Let A, B, C be objects in the category, and let $f : B \to C$, then we wish to show that $(f \times \mathrm{id}_A)\beta_{A,B} = \beta_{A,C}(\mathrm{id}_A \times f)$. Both of these morphisms are arrows $A \times B \to C \times A$ and so we need only check that they have the universal properties given below-right to show equality.



It is a simple matter to check that $\pi_C(f \times \mathrm{id}_A)\beta_{A,B} = f\pi_A = \pi_C\beta_{A,C}(\mathrm{id}_A \times f)$ and similarly that $\pi_A(f \times \mathrm{id}_A)\beta_{A,B} = \pi_B = \pi_A\beta_{A,C}(\mathrm{id}_A \times f)$, thus β is natural in its second argument.

That the hexagonal diagrams commute is nothing more than a drawn-out exercise in universal property arguments. By drawing out the large commutative diagram for $g = (\mathrm{id}_B \times \beta_{A,C})\alpha_{B,A,C}(\beta_{A,B} \times \mathrm{id}_C)$ we see that $g : (A \times B) \times C \to B \times (C \times A)$ is characterised through the universal properties $\pi_B g = \pi_B \pi_{A\times B}$ and $\pi_{C\times A} g = \beta_{A,C} v$ where $v : (A \times B) \times C \to A \times C$ is the unique map with $\pi_A v = \pi_A \pi_{A\times B}$, $\pi_C v = \pi_C$.

Should we perform a similar exercise for $h = \alpha_{B,C,A}\beta_{A,B\times C}\alpha_{A,B,C}$ we see that $\pi_B h = \pi_B \pi_{A\times B}$ as desired, but this time we have that $\pi_{C\times A} h = v'\beta_{A,B\times C}\alpha_{A,B,C}$ where $v' : (B \times C) \times A \to C \times A$ is the unique map with $\pi_A v' = \pi_A$, $\pi_C v' = \pi_C \pi_{B\times C}$.

In order to show equality between $\beta_{A,C} v$ and $v'\beta_{A,B\times C}\alpha_{A,B,C}$ as parallel morphisms $(A \times B) \times C \to C \times A$, we turn to a final universal property argument.

$$\begin{array}{ccccc} C & \longleftarrow & (A \times B) \times C & \longrightarrow & A \times B \\ \| & & \downarrow{\scriptstyle w} & & \downarrow \\ C & \longleftarrow & C \times A & \longrightarrow & A \end{array}$$

Using the commuting diagram above, we check that

$$\begin{aligned}
&\pi_A \beta_{A,C} v = \pi_A v = \pi_A \pi_{A\times B} \\
&\pi_B \beta_{A,C} v = \pi_C v = \pi_C \\
&\pi_A v'\beta_{A,B\times C}\alpha_{A,B,C} = \pi_A \beta_{A,B\times C}\alpha_{A,B,C} = \pi_A \alpha_{A,B,C} = \pi_A \pi_{A\times B} \\
&\pi_C v'\beta_{A,B\times C}\alpha_{A,B,C} = \pi_C \pi_{B\times C}\beta_{A,B\times C}\alpha_{A,B,C} = \pi_C \pi_{B\times C}\alpha_{A,B,C} = \pi_C
\end{aligned}$$

thus completing the proof. ■

As in the case of monoidal categories, we also briefly discuss the behaviour of functors which preserve the braided nature of their categories.

Def. (II) 1.1.5. A braided monoidal functor $F : \mathfrak{C} \to \mathfrak{D}$ is a monoidal functor between the monoidal categories for which the following diagram commutes for all $A, B \in \mathrm{Obj}\,\mathfrak{C}$.

$$\begin{array}{ccc} FAFB & \xrightarrow{\beta_{FA,FB}} & FBFA \\ {\scriptstyle \phi_{A,B}}\downarrow & & \downarrow{\scriptstyle \phi_{B,A}} \\ F(AB) & \xrightarrow[F\beta_{A,B}]{} & F(BA) \end{array}$$

Remark (II) 1.1.6. Natural transformations between braided monoidal functors are monoidal natural transforms between the monoidal functors, and are not required to satisfy any additional properties. ◃

Ultimately, however, our interest in braided monoidal categories is constrained to the case of symmetry.

Def. (II) 1.1.7. A braided monoidal category is called symmetric when the braiding has $\beta^2 = \mathrm{id}$.

Remark (II) 1.1.8. Symmetric monoidal functors are simply braided monoidal functors where the braiding happens to be a symmetry. ◃

As is no doubt already manifestly evident,

Prop. (II) 1.1.9. *Every cartesian monoidal category with the canonical braiding is symmetric.*

1.2. Closed monoidal categories

Remark (II) 1.2.1. This section will frequently make use of various properties of adjunctions. For the convenience of the reader, the requisite supporting theory has been exhibited in section 3. ◃

Not for the last time, we look to Set as a cartesian monoidal category for more interesting structural properties which we may take to the general monoidal case. The facet that catches our eye this time is that if A and B are sets, then $\mathrm{Set}(A,B) \in \mathrm{Obj}\,\mathrm{Set}$. In some ways, this is the result of a special privilege enjoyed by the somewhat central role of Set in the standard theory. However, a more careful treatment of this property is desirable.

In order to effect this, we must first essay the 'categorification' of our understanding of sets $\mathrm{Set}(B,C)$. Classically, we would write such a set as C^B and we would be quick to note that we have a canonical morphism $\mathrm{ev} : C^B \times B \to C$ as defined by $\mathrm{ev}(f,b) = f(b)$. Upon careful inspection, we may surmise that C^B and ev are universal with respect to the following property.

Prop. (II) 1.2.2. *For all sets A, and set functions $f : A \times B \to C$ there exists a unique set function $\lambda f : A \to C^B$ such that* $\mathrm{ev}\langle \lambda f, \mathrm{id}_B\rangle = f$.

Proof. Given f we define $(\lambda f)(a)(b) = f(a,b)$. It is a simple matter to verify that $\mathrm{ev}\langle \lambda f, \mathrm{id}_B\rangle = f$ and uniqueness follows from pointwise agreement. ■

Should we inspect the above with an eye to a more general theory, we note that the universal property is telling us something about the functor $\times B$. In particular, we may recognise that (C^B, ev) is somehow a universal arrow from $\times B$ to C. That is, for every $(A, f : A \times B \to C)$ we have a unique morphism $A \to C^B$ such that the appropriate diagram commutes. Moreover, we know that if every object has a universal arrow, as is the case in Set for $\times B$, then we have an adjunction! Thus, in one broad and permeating stroke we generalise as much as seems reasonable and define the following.

Def. (II) 1.2.3. A right-closed monoidal category is a monoidal category $\mathfrak{C}$ wherein for every object $C \in \mathrm{Obj}\,\mathfrak{C}$ the functor $\bullet \otimes C$ has a right adjoint $[C, \bullet]$. That is, for every $A, B, C \in \mathrm{Obj}\,\mathfrak{C}$, $\mathfrak{C}(A \otimes B, C) \cong \mathfrak{C}(A, [B, C])$ naturally in A, C. The image of the functor $[C, \bullet]$ is called the internal morphism object. Similarly, a left-closed monoidal category is one in which the functor $C \otimes \bullet$ has a right adjoint $[\bullet, C]$.

Remark (II) 1.2.4. Drawing upon the previous section, we immediately see that if the monoidal category is braided, then it is left-closed iff it is right-closed iff it is biclosed and so we simply say that it is a closed braided monoidal category. Importantly, in this case the isomorphism of (external) morphism objects is natural in *all* arguments. ◃

To demonstrate that we have indeed generalised the classical theory of SET, and the extent to which internal morphism objects behave as though they were actually collections of morphisms, we begin with the following, perhaps presumptuous, definition.

Def. (II) 1.2.5. In a right-closed monoidal category, we

- say that a morphism $a : I \to A$ is called a point of A.
- define $\mathrm{ev}_{A,B} : [A,B] \otimes A \to A$ to be the counit of the adjunction on $\otimes A$, so named as it plays the role of internal evaluation,
- and define $\circ_{A,B,C} : [B,C] \otimes [A,B] \to [A,C]$ to be the image of the morphism

$$\mathrm{ev}_{B,C}(\mathrm{id}_{[B,C]} \otimes \mathrm{ev}_{A,B})\alpha_{[B,C],[A,B],A} : ([B,C] \otimes [A,B]) \otimes A \to C$$

  under the adjunction isomorphism $\mathfrak{C}(([B,C]\otimes[A,B])\otimes A, C) \to \mathfrak{C}([B,C]\otimes[A,B],[A,C])$, so named as it plays the role of internal composition.

In a right-closed monoidal category, we have $\mathfrak{C}(A,B) \cong \mathfrak{C}(I \otimes A, B) \cong \mathfrak{C}(I,[A,B])$, where the first isomorphism is $\mathfrak{C}(\lambda_A, B)$, and so we can identify arrows $f : A \to B$ with points $[f] : I \to [A,B]$ of internal morphism objects. That is, morphisms from I to the internal morphism object are simply elements of the external morphism object. For the first time then, the reader may disregard the insistence of the author and see for himself that there is a sense in which we should regard morphisms from I to an object as generalised elements (*points*) of the object in question.

With this language established, in the following two propositions, we examine the interplay between external and internal morphism objects and the properties of internal composition.

Prop. (II) 1.2.6. *In a right-closed monoidal category $\mathfrak{C}$, for all objects A, B, C,*

1. *the adjoint isomorphism $\mathfrak{C}(A,[B,C]) \to \mathfrak{C}(A \otimes B, C)$ takes $f : A \to [B,C]$ to $\mathrm{ev}_{B,C}(f \otimes \mathrm{id}_B)$*
2. *for all $f : A \to B$, $\mathrm{ev}_{A,B}([f] \otimes \mathrm{id}_A) \cong f$*
3. *let $a : I \to A$ be a point, and $f : A \to B$, then $\mathrm{ev}_{A,B}([f] \otimes a) \cong fa$, a point of B*
4. $\mathrm{ev}_{B,C}\left(\mathrm{id}_{[B,C]} \otimes \mathrm{ev}_{A,B}\right)\alpha_{[B,C],[A,B],A} = \mathrm{ev}_{A,C}\left(\circ_{A,B,C} \otimes \mathrm{id}_A\right)$

Proof. We immediately recognise (1) as a trivial consequence of the definition of adjunctions in terms of units and counits. To elaborate the point, if we write the adjoint isomorphism as $\phi_{A,C} : \mathfrak{C}(A,[B,C]) \to \mathfrak{C}(A \otimes B, C)$ then we know (prop. (I) 3.0.2) that it can be stated in terms of the counit and left adjoint functor as $\phi_{A,C} = \mathrm{ev}_{B,C}(- \otimes \mathrm{id}_A)$.

That (2) holds follows from the fact that $\mathrm{ev}_{A,B}([f] \otimes \mathrm{id}_A) = f\lambda_A$, by the definition of $[f]$ and (1). For (3), $\mathrm{ev}_{A,B}([f] \otimes a) = \mathrm{ev}_{A,B}([f] \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes a) = f\lambda_A(\mathrm{id}_I \otimes a) = fa\lambda_I$ by (2) and naturality of λ. Finally, (4) follows again through a simple combination of definition and (1). ■

Prop. (II) 1.2.7. *In a right-closed monoidal category $\mathfrak{C}$, for all objects A, B, C, D*

1. *Let $f : A \to B$ and $g : B \to C$ then $\circ_{A,B,C}([g] \otimes [f]) \cong [gf]$*

2. $\circ_{A,B,D}(\circ_{B,C,D} \otimes \mathrm{id}_{[A,B]}) = \circ_{A,C,D}(\mathrm{id}_{[C,D]} \otimes \circ_{A,B,C})\alpha_{[C,D],[B,C],[A,B]}$
as morphisms $([C,D] \otimes [B,C]) \otimes [A,B] \to [A,D]$ – composition is associative within the monoidal structure

3. $\circ_{A,B,B}([\mathrm{id}_B] \otimes \mathrm{id}_{[A,B]}) = \lambda_{[A,B]}$ *and* $\circ_{A,A,B}(\mathrm{id}_{[A,B]} \otimes [\mathrm{id}_A]) = \rho_{[A,B]}$ *– composition is unital within the monoidal structure*

4. $[A \otimes B, C] \cong [A,[B,C]]$

Proof. Let us begin by writing ${\phi^{-1}}_{A,C} : \mathfrak{C}(AB, C) \to \mathfrak{C}(A,[B,C])$ for the adjunction isomorphism, natural in both arguments. The statement of left-hand side of (1) then becomes ${\phi^{-1}}_{[B,C][A,B],C}(\mathrm{ev}_{B,C}(\mathrm{id}_{[B,C]} \otimes \mathrm{ev}_{A,B})\alpha_{[B,C],[A,B],A})([g] \otimes [f])$. However, ϕ^{-1} is natural and so we turn to the appropriate naturality square to proceed with simplification.

$$
\begin{array}{ccc}
\mathfrak{C}\big(([B,C][A,B])A, C\big) & \xrightarrow{\ {\phi^{-1}}_{[B,C][A,B],C}\ } & \mathfrak{C}([B,C][A,B],[A,C]) \\
{\scriptstyle \mathfrak{C}\left(([g]\otimes[f])A,\,C\right)}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{C}([g]\otimes[f],\,C)} \\
\mathfrak{C}((II)A, C) & \xrightarrow[\ {\phi^{-1}}_{II,C}\ ]{} & \mathfrak{C}(II,[A,C])
\end{array}
$$

Thus, we have the equalities

$$
\begin{aligned}
&{\phi^{-1}}_{[B,C][A,B],C}(\mathrm{ev}_{B,C}([B,C]\,\mathrm{ev}_{A,B})\alpha_{[B,C],[A,B],A})([g] \otimes [f]) && \\
=&{\phi^{-1}}_{II,C}\Big(\mathrm{ev}_{B,C}([B,C]\,\mathrm{ev}_{A,B})\alpha_{[B,C],[A,B],A}\,(([g] \otimes [f])\,A)\Big) && \text{(naturality of } \phi^{-1}) \\
=&{\phi^{-1}}_{II,C}\,(\mathrm{ev}_{B,C}([B,C]\,\mathrm{ev}_{A,B})([g] \otimes ([f]A))\,\alpha_{I,I,A}) && \text{(naturality of } \alpha) \\
=&{\phi^{-1}}_{II,C}\,(\mathrm{ev}_{B,C}([g] \otimes f\lambda_A)\alpha_{I,I,A}) && \text{(prop. (II) 1.2.6 (2))} \\
=&{\phi^{-1}}_{II,C}\,(g\lambda_B(I\,(f\lambda_A))\alpha_{I,I,A}) && \text{(prop. (II) 1.2.6 (2))} \\
=&{\phi^{-1}}_{II,C}\,(gf\lambda_A\,(I\lambda_A)\,\alpha_{I,I,A}) && \text{(naturality of } \lambda) \\
=&{\phi^{-1}}_{II,C}\,(gf\lambda_A\,(\rho_I A)) && \text{(def. (II) 1.0.1, triangle)} \\
=&{\phi^{-1}}_{II,C}\,(gf\lambda_A\,(\lambda_I A)) && \text{(prop. (II) 1.0.2 (3))}
\end{aligned}
$$

In order to progress from here, we must introduce an isomorhpism which will allow us to recast the above into $[gf]$. We carefully note that the naturality of ϕ^{-1} gives us $\mathfrak{C}\,({\lambda^{-1}}_I,[A,C])\,{\phi^{-1}}_{II,C} = {\phi^{-1}}_{I,C}\mathfrak{C}\,({\lambda^{-1}}_I A, C)$. Using this we may finally state

$$
{\phi^{-1}}_{II,C}\,(gf\lambda_A\,(\lambda_I A))\,{\lambda^{-1}}_I\lambda_I = {\phi^{-1}}_{I,C}\,(gf\lambda_A)\,\lambda_I = [gf]\lambda_I,
$$

thereby completing the proof of (1).

When considering (2), we first note that we expect both the arrows, via adjunction, to have the form $(([C,D]\otimes[B,C])\otimes[A,B])\otimes A \to D$. Of course, we know that we cannot use ev – essentially the content of the adjoint to internal composition – when the domain is in such a form. Given this, we expect the adjoints of the arrows in (2) to change the domain $(([C,D]\otimes[B,C])\otimes[A,B])\otimes A \to [C,D]\otimes([B,C]\otimes([A,B]\otimes A))$, via some isomorphism.

With this context established, we claim that the following pairs are adjoint to one another.

$$\mathrm{adj}\quad \begin{cases} \circ_{A,B,D}\,(\circ_{B,C,D}\otimes \mathrm{id}_{[A,B]}) \\ \circ_{A,C,D}\,(\mathrm{id}_{[C,D]}\otimes\circ_{A,B,C})\alpha_{[C,D],[B,C],[A,B]} \end{cases} \longleftrightarrow \begin{cases} \kappa\alpha_{[C,D],[B,C],[A,B]\otimes A}\alpha_{[C,D]\otimes[B,C],[A,B],A} \\ \kappa(\mathrm{id}_{[C,D]}\otimes\alpha_{[B,C],[A,B],[A]})\alpha_{[C,D],[B,C]\otimes[A,B],A}(\alpha_{[C,D],[B,C],[A,B]}\otimes \mathrm{id}_A) \end{cases}$$

where $\kappa = \mathrm{ev}_{C,D}(\mathrm{id}_{[C,D]}\otimes\mathrm{ev}_{B,C}(\mathrm{id}_{[B,C]}\otimes\mathrm{ev}_{A,B}))$. Were this to be the case, it would immediately follow, by the pentagonal identity (def. (II) 1.0.1), that the bottom pair were equal and consequently that the top pair were equal.

Thus, in order to prove (2) we must simply prove that we have the isomorphism proposed above. The process is fairly mechanical, and revolves around the naturality of α and ϕ, and makes use of prop. (II) 1.2.6 (4) once in each case. We will show only the first case, and leave the second to the capable hands of the reader.

$$\circ_{A,B,D}\,(\circ_{B,C,D}[B,A])$$

(defn.)

$$= \phi^{-1}{}_{[B,D][A,B],D}\left[\mathrm{ev}_{B,D}\,([B,D]\,\mathrm{ev}_{A,B})\,\alpha_{[B,D],[A,B],A}\right](\circ_{B,C,D}[A,B])$$

(naturality of ϕ^{-1})

$$= \phi^{-1}{}_{([C,D][B,C])[A,B],D}\left[\mathrm{ev}_{B,D}\,([B,D]\,\mathrm{ev}_{A,B})\,\alpha_{[B,D],[A,B],A}\,((\circ_{B,C,D}[A,B])A)\right]$$

(naturality of α, we omit subscripts on ϕ^{-1})

$$= \phi^{-1}\left[\mathrm{ev}_{B,D}\,([B,D]\,\mathrm{ev}_{A,B})\,(\circ_{B,C,D}\,([A,B]A))\,\alpha_{[C,D][B,C],[A,B],A}\right]$$

(rearrange composite)

$$= \phi^{-1}\left[\mathrm{ev}_{B,D}\,(\circ_{B,C,D}B)\,(([C,D][B,C])\,\mathrm{ev}_{A,B})\,\alpha_{[C,D][D,C],[A,B],A}\right]$$

(prop. (II) 1.2.6 (4))

$$= \phi^{-1}\left[\mathrm{ev}_{[C,D]}\,([C,D]\,\mathrm{ev}_{B,C})\,\alpha_{[C,D],[B,C],B}\,(([C,D][B,C])\,\mathrm{ev}_{A,B})\,\alpha_{[C,D][D,C],[A,B],A}\right]$$

(naturality of α)

$$= \phi^{-1}\Big[\underbrace{\mathrm{ev}_{[C,D]}\,([C,D]\,\mathrm{ev}_{B,C})\,([C,D]\,([B,C]\,\mathrm{ev}_{A,B}))}_{\kappa}\,\alpha_{[C,D],[B,C],[A,B]A}\,\alpha_{[C,D][D,C],[A,B],A}\Big]$$

The proof of (3) is a similar mechanical exercise, except that we must make use of prop. (II) 1.0.2 (2) or the triangular identity of def. (II) 1.0.1 in order to conclude it.

$$\circ_{A,B,B}([\mathrm{id}_B][A,B]) = \phi^{-1}{}_{[B,B][A,B],B}\Big[\mathrm{ev}_{B,B}([B,B]\,\mathrm{ev}_{A,B})\,\alpha_{[B,B],[A,B],A}\Big]([\mathrm{id}_B][A,B])$$

(naturality of ϕ^{-1})

$$= \phi^{-1}{}_{I[A,B],B}\Big[\mathrm{ev}_{B,B}([B,B]\,\mathrm{ev}_{A,B})\,\alpha_{[B,B],[A,B],A}(([\mathrm{id}_B][A,B])A)\Big]$$

(naturality of α)

$$= \phi^{-1}\Big[\mathrm{ev}_{B,B}([B,B]\,\mathrm{ev}_{A,B})([\mathrm{id}_B]([A,B]A))\alpha_{I,[A,B],A}\Big]$$

(prop. (II) 1.2.6 (2))

$$= \phi^{-1}\Big[\lambda_B\Big(I\big(\underbrace{\mathrm{ev}_{A,B}([A,B]A)}\big)\Big)\alpha_{I,[A,B],A}\Big]$$

($\mathrm{ev}_{A,B}([A,B]\,\mathrm{id}_A) = \mathrm{ev}_{A,B}$ by prop. (I) 3.0.2 and naturality of ϕ)

$$= \phi^{-1}\Big[\lambda_B(I\,\mathrm{ev}_{A,B})\,\alpha_{I,[A,B],A}\Big]$$

(naturality of λ)

$$= \phi^{-1}\Big[\mathrm{ev}_{A,B}\,\lambda_{[A,B]A}\alpha_{I,[A,B],A}\Big]$$

(prop. (II) 1.0.2 (2))

$$= \phi^{-1}\Big[\mathrm{ev}_{A,B}(\lambda_{[A,B]}A)\Big]$$

(prop. (II) 1.2.6 (1))

$$= \lambda_{[A,B]}$$

Again, we give only one of the two equalities, as the other proof is entirely similar – note that instead of requiring prop. (II) 1.0.2 (2) to tie matters together, the other equality makes use of the triangular identity of def. (II) 1.0.1.

Finally, for (4), let $\tau_{-,A,B} = \mathfrak{C}(\alpha^{-1}{}_{-,A,B}, C)$ to find

$$h_{[A,[B,C]]} = \mathfrak{C}(-,[A,[B,C]]) \overset{\mathrm{adj}}{\cong} \mathfrak{C}(-\otimes A,[B,C]) \overset{\mathrm{adj}}{\cong} \mathfrak{C}((-\otimes A)\otimes B, C)$$
$$\overset{\tau}{\cong} \mathfrak{C}(-\otimes(A\otimes B), C) \overset{\mathrm{adj}}{\cong} \mathfrak{C}(-,[A\otimes B, C]) = h_{[A\otimes B,C]}$$

Thus, by Yoneda, $[A\otimes B, C]\cong[A,[B,C]]$. ■

In summary then, a right-closed monoidal category affords a rich internal structure. There are internal morphism objects with a well-defined, unital and associative composition law, and the internal morphism objects support the 'same' adjunction formula as do the external ones. Indeed, it would appear that we could reformulate many statements in the general theory to those about *internal* morphisms in some right-closed monoidal category without any loss of generality.

That Cat is closed monoidal affords us an alternative view of the Yoneda lemma. If we let $h_- : \mathfrak{C} \to [\mathfrak{C}^{\mathrm{op}}, \textsc{Set}]$ be the Yoneda embedding, then

Prop. (II) 1.2.8. *h_- as the image of the functor $\mathfrak{C}(-,-) : \mathfrak{C}^{\mathrm{op}}\times\mathfrak{C} \to Set$ under the adjunction* $\textsc{Cat}(\mathfrak{C},[\mathfrak{C}^{\mathrm{op}},\textsc{Set}]) \cong \textsc{Cat}(\mathfrak{C}^{\mathrm{op}}\times\mathfrak{C},\textsc{Set})$.

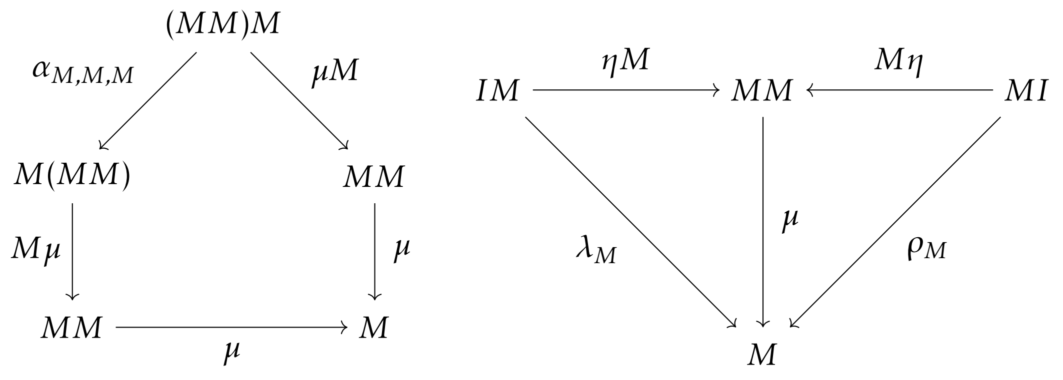
1.3. Monoids in monoidal categories

Now that we are satisfied with the existence of morphisms between monoidal categories, we see that we may form the category of (small) monoidal categories. The reader is encouraged to convince himself that the monoidal nature of the objects and morphisms in this category does not, in any way, prohibit the category from supporting all finite products (in the same manner as does Cat). Thus, among other possible structures, we may endow that category of monoidal categories with a cartesian monoidal structure.

As was pointed out earlier, we think of morphisms from the identity object of a monoidal category to a given object as generalised elements of that object. Ergo, a 'generalised object' of a monoidal category ought to correspond to monoidal functors from the terminal category 1 to the monoidal category in question, as morphisms within the cartesian monoidal category of monoidal categories.

Recall that 1 is the category with only one object and only the identity morphism, and is endowed with a monoidal structure in the obvious and trivial manner. Then, to give a functor from 1 to $\mathfrak{C}$ is to give an object of $\mathfrak{C}$, say $F\star = M$. We are then afforded morphisms $\phi_{\star,\star} : MM \to M$ and $\varepsilon : I \to M$. These are suspiciously reminiscent of multiplication and identity operations in a monoid. Indeed, should we examine the diagrams in def. (II) 1.0.7 carefully, we find we are able to make the following general definition.

Def. (II) 1.3.1. A monoid in a monoidal category $\mathfrak{C}$ is given by an object $M \in \mathrm{Obj}\,\mathfrak{C}$ equipped with morphisms $\mu : MM \to M$ and $\eta : I \to M$, known as the multiplication and unit respectively, such that the following diagrams commute.

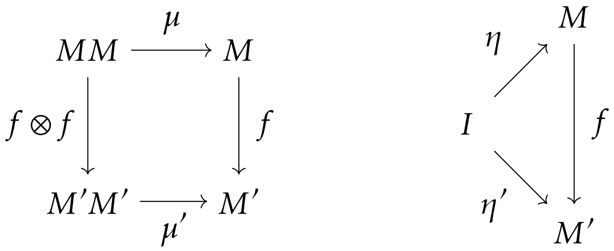


Def. (II) 1.3.2. If the monoidal category has a symmetry β then the monoid is said to be commutative if $\mu\beta = \mu$.

In the above, the left diagram expresses the associativity of the monoidal multiplication while the right expresses the right and left identity laws. It goes without saying that a monoid in the cartesian monoidal category Set is just a monoid in the usual sense.

If monoids are monoidal functors from the terminal category, then we may guess that monoid morphisms, whatever those may be, are monoidal natural transforms between such functors. Indeed, generalising the resulting diagrams and requirements we define monoid morphisms as follows.

Def. (II) 1.3.3. If (M,μ,η) and (M',μ',η') are monoids in the same monoidal category then a morphism $f : M \to M'$ is a morphism of monoids if the following diagrams commute.



Here, the left diagram makes explicit that multiplication and the morphism should commute, while the right diagram enforces that the morphism take the identity to the identity. As such, in Set, this is simply the definition of a monoid homomorphism. As a matter of course, one is led to consider the category of monoids on a monoidal category, $\mathrm{Mon}\,\mathfrak{C}$. Given our inspiration for defining monoids, it should come as no surprise that

Prop. (II) 1.3.4. *For any monoidal category* $\mathfrak{C}$, $\mathrm{Mon}\,\mathfrak{C} \cong [1,\mathfrak{C}]$.

Proof. Consider the maps $\alpha : \mathrm{Mon}\,\mathfrak{C} \to [1,\mathfrak{C}]$ and $\beta : [1,\mathfrak{C}] \to \mathrm{Mon}\,\mathfrak{C}$ as defined by $\alpha(M,\mu,\eta) = ([M],[\mu],\eta)$ and $\beta(F,\phi,\varepsilon) = (F\star,\phi_{\star,\star},\varepsilon)$ on objects, and $\alpha(f) = [f]$ and $\beta(\tau) = \tau_\star$ on arrows. Here we have used the functor $[M]$ where $[M](\star) = M$ and $[M](f) = \mathrm{id}_M$, and the natural transformation $[f]$ which is the constant f natural transformation. It is simple to verify that these maps are indeed functors and inverse to one another, and that they have domain and codomain as stated. ■

Prop. (II) 1.3.5. *Let* $(\mathfrak{C},I_\mathfrak{C})$ *and* $(\mathfrak{D},I_\mathfrak{D})$ *be monoidal categories, and* $F : \mathfrak{C} \to \mathfrak{D}$ *be a lax monoidal functor between them. The image of a monoid* (M,μ_M,η_M) *in* $\mathfrak{C}$ *has an induced monoid structure in* $\mathfrak{D}$*. Moreover, such a functor takes monoid morphisms to monoid morphisms in this sense.*

Proof. We shall prove that $(FM, \mu_{FM}, \eta_{FM})$ is a monoid in $\mathfrak{D}$, where we define the arrows $\eta_{FM} = F\eta_M\varepsilon : I_{\mathfrak{D}} \to FM$ and $\mu_{FM} = F\mu_M\phi_{M,M} : FM \otimes FM \to FM$.

We begin with the unitality diagrams. Consider that we wish to demonstrate that $\lambda_{\mathfrak{D}} = \mu_{FM}(\eta_{FM} \otimes \mathrm{id}_{FM})$. To do so, we make use of the diagram for $\lambda_{\mathfrak{D}}$ in def. (II) 1.0.7 which tells us that $\lambda_{\mathfrak{D}} = F\lambda_{\mathfrak{C}}\phi_{I_{\mathfrak{C}},M}(\varepsilon \otimes \mathrm{id}_{FM})$. However, as M was a monoid in $\mathfrak{C}$, we must have that $F\lambda_{\mathfrak{C}} = F\mu_M F(\eta_M \otimes \mathrm{id}_M)$. Now, by naturality of ϕ we have $F(\eta_M \otimes \mathrm{id}_M)\phi_{I_{\mathfrak{C}},M} = \phi_{M,M}(F\eta_M \otimes \mathrm{id}_{FM})$ and thus

$$\begin{aligned}
\lambda_{\mathfrak{D}} &= F\lambda_{\mathfrak{C}}\phi_{I_{\mathfrak{C}},M}(\varepsilon \otimes \mathrm{id}_{FM}) \\
&= F\mu_M F(\eta_M \otimes \mathrm{id}_M)\phi_{I_{\mathfrak{C}},M}(\varepsilon \otimes \mathrm{id}_{FM}) && \text{(monoid in } \mathfrak{C}\text{)} \\
&= F\mu_M\phi_{M,M}(F\eta_M \otimes \mathrm{id}_{FM})(\varepsilon \otimes \mathrm{id}_{FM}) && \text{(naturality of } \phi\text{)} \\
&= \mu_{FM}\eta_{FM} && \text{(definition)}
\end{aligned}$$

The proof follows, *mutatis mutandis*, for $\rho_{\mathfrak{D}}$.

To see that the associativity holds in $\mathfrak{D}$, we must turn to the associativity diagram for ϕ in def. (II) 1.0.7. Should we paste the image of the associativity diagram for M in $\mathfrak{C}$ to the bottom of that diagram, we find a large commuting diagram allowing two distinct avenues of traversal. In the first case we find

$$\begin{aligned}
&F\mu_M F(\mu_M \otimes \mathrm{id}_M)\phi_{M\otimes M,M}(\phi_{M,M} \otimes \mathrm{id}_{FM}) \\
=&(F\mu_M\phi_{M,M})(F\mu_M \otimes \mathrm{id}_{FM})(\phi_{M,M} \otimes \mathrm{id}_{FM}) && \text{(naturality of } \phi\text{)} \\
=&\mu_{FM}(\mu_{FM} \otimes \mathrm{id}_{FM}),
\end{aligned}$$

whereas the second gives

$$\begin{aligned}
&F\mu_M F(\mathrm{id}_M \otimes \mu_M)\phi_{M,M\otimes M}(\mathrm{id}_{FM} \otimes \phi_{M,M})\alpha_{\mathfrak{D}} \\
=&(F\mu_M\phi_{M,M})(F\,\mathrm{id}_M \otimes F\mu_M)(\mathrm{id}_{FM} \otimes \phi_{M,M})\alpha_{\mathfrak{D}} && \text{(naturality of } \phi\text{)} \\
=&\mu_{FM}(\mathrm{id}_{FM} \otimes \mu_{FM})\alpha_{\mathfrak{D}}.
\end{aligned}$$

As the large diagram commutes, these two are equal and therefore $(FM, \mu_{FM}, \eta_{FM})$ is a monoid in $\mathfrak{D}$.

Finally, if $f : M \to N$ is a morphism of monoids (M, μ_M, η_M) and (N, μ_N, η_N) in $\mathfrak{C}$, then we seek to show that Ff is a morphism of monoids in $\mathfrak{D}$. Consider that $Ff\eta_{FM} = FfF\eta_M\varepsilon = F(f\eta_M)\varepsilon = F\eta_N\varepsilon = \eta_{FN}$ and that

$$\begin{aligned}
Ff\mu_{FM} &= FfF\mu_M\phi_{M,M} = F(f\mu_M)\phi_{M,M} \\
&= F(\mu_N(f \otimes f))\phi_{M,M} && \text{(monoid morphism in } \mathfrak{C}\text{)} \\
&= F\mu_N F(f \otimes f)\phi_{M,M} = F\mu_N\phi_{N,N}(Ff \otimes Ff) && \text{(naturality of } \phi\text{)} \\
&= \mu_{FN}(Ff \otimes Ff),
\end{aligned}$$

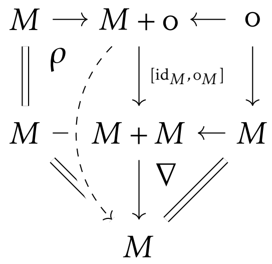
thereby completing the proof. ∎

Cor. (II) 1.3.6. *A lax monoidal functor* $F : \mathfrak{C} \to \mathfrak{D}$ *induces a functor* $\mathrm{Mon}\,\mathfrak{C} \to \mathrm{Mon}\,\mathfrak{D}$.

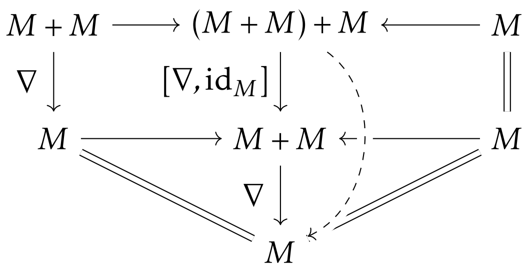
Prop. (II) 1.3.7. *In a cocartesian monoidal category, every object admits a unique commutative monoid structure, and morphisms between objects are morphisms of the induced monoids.*

Proof. Let $\mathfrak{C}$ be have all finite coproducts, fix $M \in \mathrm{Obj}\,\mathfrak{C}$ and consider $M+M$ in the monoidal category $(\mathfrak{C}, +, 0)$. We obviously have morphisms $\nabla : M+M \to M$ (known as the codiagonal) and $0_M : 0 \to M$ and so only need to show that the diagrams in def. (II) 1.3.1 commute.

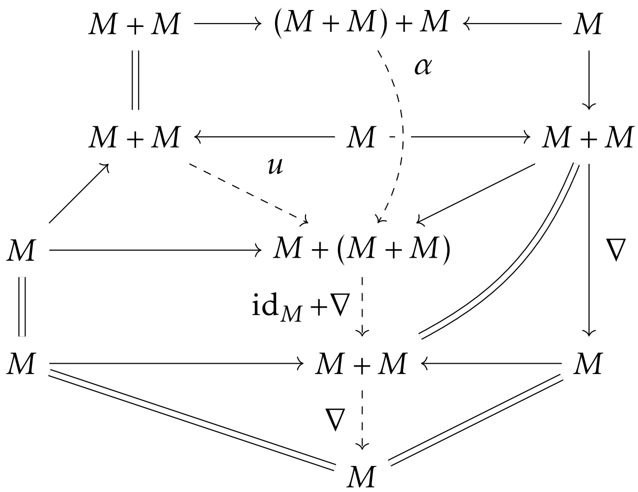
We address first the comparatively short matter of unitality. As is likely evident, that the requisite diagrams commute is due to a simple universal property argument. In particular, to see that $\nabla[\mathrm{id}_M, 0_M] = \rho_M$ consider the following commutative diagram.



The above argument applies, *mutatis mutandis*, to λ. The matter of associativity is more nuanced, but is still simply a universal property argument. In particular, we wish to show that $\nabla[\mathrm{id}_M, \nabla]\alpha = [\nabla, \mathrm{id}_M]\nabla$. Drawing out the appropriate diagram for the right-hand side, we find that it is characterised by the following universal property in the commutative diagram below.



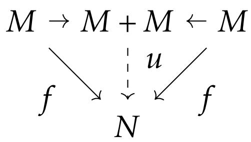
Thus we must verify that $f = \nabla[\mathrm{id}_M, \nabla]\alpha$ has $f\iota_M = \mathrm{id}_M$ and $f\iota_{M+M} = \nabla$ to show the required equality. To see that it does, we draw the diagram giving α, connected appropriately to those for $[\mathrm{id}_M, \nabla]$ and ∇, below. The result follows by noting that $f\iota_M = \nabla\iota_M = \mathrm{id}_M$ (right edge) and $\nabla[\mathrm{id}_M, \nabla]\alpha\iota_{M+M} = \nabla[\mathrm{id}_M, \nabla]u = \nabla\,\mathrm{id}_M$.



Then, that the monoid is commutative is trivial by universal properties again as we know that the symmetry $\beta : M+M \to M+M$ satisfies $\beta\iota_M = \iota_M$ and so $\nabla\beta\iota_M = \iota_M$ gives us $\nabla\beta = \nabla$ by universal property.

To see that the monoid is unique, suppose $\mu : M+M \to M$ was another arrow such that (M, μ, o_M) formed a monoid (o_M is obviously unique). As such, it must be the case that $\lambda_M = \mu[o_M, \mathrm{id}_M]$. However, $\lambda_M\iota_M = \mathrm{id}_M$ and so $\mu[o_M, \mathrm{id}_M]\iota_M = \mu_M\iota_M = \mathrm{id}_M$. Thus, by the universal property of ∇, $\mu = \nabla$.

Finally, suppose $f : M \to N$ was an arrow in the category. It is a trivial matter to see that $f o_M = o_N$ and so we must only check that $f\nabla = \nabla[f, f]$. Once more we apply a standard universal property argument.



Using the above commuting diagram, we see that we have two potential candidates for u, viz., $f\nabla$ and $\nabla[f, f]$. By construction, $f\nabla\iota_M = f$ and it is a simple matter to verify that $\nabla[f, f]\iota_M = f$, thus f is a morphism of monoids. ∎

To do This is probably false, or else needs to be checked for a monoidal isomorphism. (9)

Cor. (II) 1.3.8. *If $\mathfrak{C}$ is a cocartesian monoidal category, then $\mathfrak{C} \cong \mathrm{Mon}\,\mathfrak{C}$.*

2. Category Objects

In this section we attempt to realise a generalisation of categories by viewing them as objects internal to other categories. Said another way, we have (at least implicitly) appealed to the notion that $\operatorname{Obj}\mathfrak{C}$ and $\operatorname{Mor}\mathfrak{C}$ form classes, but nothing more interesting beyond. Indeed, the careful reader may wonder why it is that these collections cannot demonstrate more interesting and general behaviour themselves. To realise such a theory, we begin by introducing the notion of a precategory.

Def. (II) 2.0.1. Given a category $\mathfrak{C}$, a precategory in $\mathfrak{C}$ is a pair of objects (A, J) with two morphisms $\partial_0, \partial_1 : A \rightrightarrows J$. A morphism of precategories $F : (A, J) \to (A', J')$ is a pair of arrows $F_A : A \to A'$ and $F_J : J \to J'$ in $\mathfrak{C}$ such that the $F_J \partial_i = \partial'_i F_A$ for $i \in \{0, 1\}$. Together with the evident component-wise composition and identity, precategories in $\mathfrak{C}$ form a category $\operatorname{Precat}\mathfrak{C}$.

As should be clear, a precategory $A \rightrightarrows J$ can be thought of as comprising an arrows object A, a morphism object J and assignments $\operatorname{dom}, \operatorname{cod} : A \rightrightarrows J$. Note well here that precategories capture only the global data inherent to a category, that is, a precategory is not explicitly equipped with any notion of composition as this is this is specific to a given pair of morphisms (though there is a suitable notion of *composability*) or indeed any notion of identity morphisms (again this depends on the object in question). Said another way, given that we have only general, opaque objects representing the collections arrows and morphisms we do not assume any way to peer inside them and deal with specifics.

On this level then, when elevating the idea of functors to morphisms of precategories, we find that the only relevant, global data tracked by a functor is the interplay between the domains and codomains of mapped arrows, as expressed in the definition. It is then a simple matter to observe that we have an assignment $U : \textsc{Cat} \to \operatorname{Precat}\textsc{Set}$ (where the former is taken to be the category of small categories) which sends $\mathfrak{C} \mapsto \operatorname{dom}_{\mathfrak{C}}, \operatorname{cod}_{\mathfrak{C}} : \operatorname{Mor}\mathfrak{C} \rightrightarrows \operatorname{Obj}\mathfrak{C}$ and functors to their object and morphism maps. That this assignment is a functor is simple to verify and amounts to the fact a functor $F : \mathfrak{C} \to \mathfrak{D}$ has $F \operatorname{dom}_{\mathfrak{C}} = \operatorname{dom}_{\mathfrak{D}} F$ by definition. Of course, the presence of such a 'forgetful' functor begs the question of the existence of free-object adjoint. We explore this briefly below.

Free small categories

To do Free small category (10)

2.1. The monoidal category J-Precat $\mathfrak{C}$

Recall that in our earlier discussion of the data captured by a precategory it was mentioned in passing that, although composition is conspicuously absent, the general notion of composability may yet be realised for such a structure. To see this, we must make an assumption about the nature of the underlying category $\mathfrak{C}$ and introduce a further definition.

Def. (II) 2.1.1. Fix an object $J \in \mathrm{Obj}\,\mathfrak{C}$ and let J-Precat $\mathfrak{C}$ be the category whose objects are precategories $(A, J, \partial_0, \partial_1 : A \rightrightarrows J)$ where J is fixed and whose morphisms are morphisms of precategories $(F_A, F_J) : (A, J) \to (A', J)$ such that $F_J = \mathrm{id}_J$.

In a suitably general (and generous) manner, J-Precat $\mathfrak{C}$ may be thought of as comprising all the possible (and impossible) morphism structures on the collection of objects J. Naturally, not all of these *could* form a category structure (for instance, if $A = \phi$) but J-Precat $\mathfrak{C}$ will form a convenient platform to describe those which do.

Among the various objects in J-Precat $\mathfrak{C}$ we always have $(J, \mathrm{id}_J, \mathrm{id}_J)$. This particular precategory can be interpreted as the categorical structure on objects J whose only arrows are identity arrows. Even though we have not made any explicit allowances for identity arrows (that is, elements of J) in general, we may still readily prove the abstract version of the statement that any parallel functors $F, G : \mathfrak{C} \rightrightarrows \mathfrak{D}$ between categories with $\mathrm{Obj}\,\mathfrak{C} = \mathrm{Obj}\,\mathfrak{D}$ where $\mathfrak{D}$ is discrete and both functors are the identity on objects, must be equal.

Prop. (II) 2.1.2. *In J-Precat $\mathfrak{C}$, any two parallel morphisms of precategories $F, G : A \rightrightarrows J$ are equal.*

Proof. Let us write $(A, \partial_0^A, \partial_1^A)$, and consider that $F = F\partial_0^J = \partial_0^A = G\partial_0^J = G$. ■

Of course, in both the case of categories and precategories above, this statement is trivial. We mention it here only because the case of functors is usually proven by elaborating the definition of a functor on specific arrows, but in the context of precategories there can be no such statement. Thus, this is suggestive of the fact that more may be extracted from opaque objects $\mathrm{Obj}\,\mathfrak{C}$ and $\mathrm{Mor}\,\mathfrak{C}$ than may initially have been suspected.

Remark (II) 2.1.3. It is important to note our proof above may have made use of ∂_1^A instead, and having done so we would have shown $\partial_0^A = \partial_1^A$. This also is the analogue of the statement that such functors $F, G : \mathfrak{C} \rightrightarrows \mathfrak{D}$ only exist when all morphisms of $\mathfrak{C}$ are endomorphisms.

Moreover, as in the functor case, in general $(J, \mathrm{id}_J, \mathrm{id}_J)$ is not terminal. To see this we only need a precategory with non-endomorphisms. For $\mathfrak{C} = \textsc{Set}$ and $J = \{0, 1\}$, $A = \{0 \to 1\}$ with $\partial_i^A(0 \to 1) = i$ suffices. ◁

Now that we are comfortable with the form of J-$\textsc{Precat}\mathfrak{C}$, we move to consider the generalisation of composability. If $\mathfrak{C}$ has pullbacks, then give two objects A, B in J-$\textsc{Precat}\mathfrak{C}$ we may form the following pullback in $\mathfrak{C}$, adopting the standard (though arguably reversed) order of function composition.

$$\begin{array}{ccc} A \times_J B & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \partial_1^B \\ A & \xrightarrow[\partial_0^A]{} & J \end{array}$$

Evidently, $A \times_J B$ bears the interpretation as those pairs of arrows (a, b) such that $\operatorname{dom} a = \operatorname{cod} b$. With this understanding and the interpretation of J above, we may expect that $A \times_J J \cong A$. Moreover, while we recognise that in general $A \times_J (B \times_J C) \neq (A \times_J B) \times_J C$, we may well anticipate that they are at least isomorphic. To the observant reader with an excellent memory, such a setup may seem suspiciously close to a familiar general structure. Indeed,

Prop. (II) 2.1.4. *If $\mathfrak{C}$ has all binary pullbacks then for any $J \in \operatorname{Obj}\mathfrak{C}$, $(J\text{-}\textsc{Precat}\mathfrak{C}, \times_J, J)$ forms a monoidal category.*

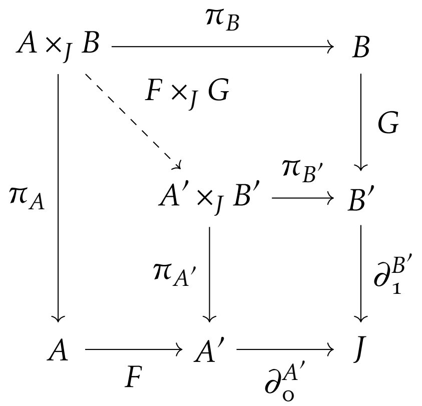
In order to prove this, however, we must establish some results of a somewhat technical nature. In particular,

Prop. (II) 2.1.5. *If $\mathfrak{C}$ has all binary pullbacks then for any $J, A, B, C \in \operatorname{Obj}\mathfrak{C}$,*

1. *$\times_J$ is a bifunctor $J\text{-}\textsc{Precat}\mathfrak{C} \times J\text{-}\textsc{Precat}\mathfrak{C} \to J\text{-}\textsc{Precat}\mathfrak{C}$*
2. *$J \times_J A \cong A \cong A \times_J J$ as J-precategories, naturally in A*
3. *$(A \times_J B) \times_J C \cong A \times_J (B \times_J C)$ as J-precategories, naturally in A, B, C*
4. *the triangle and pentagon diagrams of def. (II) 1.0.1 commute*

While none of these results are especially easy to prove, none of them are particularly enlightening. This may be seen ahead of time by noting that the only real tool capable of proving that various composites involving $\times_J$ are equal is that of universal properties and so, on this level, all of the proofs reduce to characterising various arrows and showing that two composites of interest share this characterisation. As such, we only give explicit proof for the first claim, and the rest follow by mechanical computation.

Proof (prop. (II) 2.1.5 (1)). The action on objects is given by the obvious assignment $((A, \partial_0^A, \partial_1^A), (B, \partial_0^B, \partial_1^B)) \mapsto (A \times_J B, \partial_0^{A\times_J B} = \partial_0^A \pi_A, \partial_1^{A\times_J B} = \partial_1^B \pi_B)$ and so we must extend this to arrows. Given morphisms of J-precategories $F : A \to A'$, $G : B \to B'$ we form $F \times_J G$ in the obvious manner, noting that $\partial_1^{B'} G\pi_B = \partial_1^B \pi_B = \partial_0^A \pi_A = \partial_0^{A'} F\pi_A$ by definition of the pullback and morphisms of J-precategories.



Due to the universal property construction, the functoriality of this assignment is immediate. ■

2.2. Internal categories

To do Category object (11) **To do** Internal functor (12) **To do** Internal nat as right adjoint to product? (13)

III. Enriched categories

A recurring theme in the theory of categories is that meaningful results and information can be obtained not by studying the constituents of any given object, but rather by studying the interdependence that the object has with other, related objects. That is, in a category it is the arrows that are in some sense more important than the 'elements' of a given object, should such a notion even exist. Given this, it is curious then that the collection of morphisms between two objects be defined to comprise individual elements – it has, in a way, a privileged position. Moreover, it seems at odds with the rampant generalisation present elsewhere that we should be forced to deal with categories whose collections of morphisms are confined to form sets (or classes) and not other interesting structures – groups, topological spaces, and even categories themselves! To remedy these shortcomings, we introduce the notion of enriched categories.

1. Basic notions

Def. (III) 1.0.1. Let $(\mathfrak{V},\otimes,I,\alpha,\lambda,\rho)$ be a monoidal category. A $\mathfrak{V}$-enriched category or $\mathfrak{V}$-category, $\mathfrak{C}$ is a collection of objects $\operatorname{Obj}\mathfrak{C}$ such that

1. for each ordered pair of objects $(A,B) \in \operatorname{Obj}\mathfrak{C} \times \operatorname{Obj}\mathfrak{C}$ there is an associated object $\mathfrak{C}(A,B) \in \operatorname{Obj}\mathfrak{B}$, called the morphism object from A to B
2. for each ordered triple $(A,B,C) \in \operatorname{Obj}\mathfrak{C}^{\times 3}$ there is a morphism $\circ_{A,B,C} \in \operatorname{Mor}\mathfrak{V}$ with $\circ_{A,B,C} : \mathfrak{C}(B,C)\otimes\mathfrak{C}(A,B) \to \mathfrak{C}(A,C)$, called the composition morphism
3. for each object $A \in \operatorname{Obj}\mathfrak{C}$ there is a morphism $j_A : I \to \mathfrak{C}(A,A)$, called the identity element

where the following diagrams must commute for all $A,B,C,D \in \operatorname{Obj}\mathfrak{C}$, expressing the associativity of composition and that composition is unital, respectively.

$$\begin{array}{ccccc}
(\mathfrak{C}(C,D)\otimes\mathfrak{C}(B,C))\otimes\mathfrak{C}(A,B) & & \xrightarrow{\alpha_{\mathfrak{C}(C,D),\mathfrak{C}(B,C),\mathfrak{C}(A,B)}} & & \mathfrak{C}(C,D)\otimes(\mathfrak{C}(B,C)\otimes\mathfrak{C}(A,B)) \\
{\scriptstyle \circ_{B,C,D}\otimes\mathrm{id}_{\mathfrak{C}(A,B)}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{id}_{\mathfrak{C}(C,D)}\otimes\circ_{A,B,C}} \\
\mathfrak{C}(B,D)\otimes\mathfrak{C}(A,B) & \xrightarrow[\circ_{A,B,D}]{} & \mathfrak{C}(A,D) & \xleftarrow[\circ_{A,C,D}]{} & \mathfrak{C}(C,D)\otimes\mathfrak{C}(A,C)
\end{array}$$

$$\begin{array}{ccccc}
\mathfrak{C}(B,B)\otimes\mathfrak{C}(A,B) & \xrightarrow{\circ_{A,B,B}} & \mathfrak{C}(A,B) & \xleftarrow{\circ_{A,A,B}} & \mathfrak{C}(A,B)\otimes\mathfrak{C}(A,A) \\
{\scriptstyle j_B\otimes\mathrm{id}_{\mathfrak{C}(A,B)}}\uparrow & \nearrow{\scriptstyle \lambda_{\mathfrak{C}(A,B)}} & & \nwarrow{\scriptstyle \rho_{\mathfrak{C}(A,B)}} & \uparrow{\scriptstyle \mathrm{id}_{\mathfrak{C}(A,B)}\otimes j_A} \\
I\otimes\mathfrak{C}(A,B) & & & & \mathfrak{C}(A,B)\otimes I
\end{array}$$

Perhaps the simplest, non-trivial example of a $\mathfrak{V}$-category is that of an enriched singleton set.

Example (III) 1.0.2

Let $\operatorname{Obj}\mathfrak{C} = \{\star\}$, and consider $\mathfrak{C}$ as a $\mathfrak{V}$-category for a monoidal category $\mathfrak{V}$. As such, $\mathfrak{C}(\star,\star) = M \in \operatorname{Obj}\mathfrak{V}$ is a single distinguished object with $j : I \to M$ and $\circ : M \otimes M \to M$, where the appropriate diagrams commute. Careful inspection reveals these diagrams to be precisely those present in def. (II) 1.3.1 and so an enriched singleton is precisely a monoid object in the underlying monoidal category.

To demonstrate that enriched categories are indeed a generalisation of standard categories, we note the following two cases of interest.

Example (III) 1.0.3

If we take $\mathfrak{V}$ to be the cartesian monoidal category Set, then a $\mathfrak{V}$-category can be seen simply as a locally small category. If we are more daring and set $\mathfrak{V}$ to be the cartesian monoidal category Cat of small categories, then we recover a 2-category.

Example (III) 1.0.4

Finally, with an eye to closed categories, we note that if $\mathfrak{V}$ is a right closed monoidal category then it is canonically enriched over itself, with $\mathfrak{V}(A,B)$ defined to be $[A,B]$, $j_A = [\mathrm{id}_A]$ and $\circ$ as before. That the relevant diagrams commute has already been shown in prop. (II) 1.2.7. Thus, Set, Ab, Cat are all enriched over themselves.

Def. (III) 1.0.5. A functor between $\mathfrak{V}$-categories $F : \mathfrak{C} \to \mathfrak{D}$, a $\mathfrak{V}$-functor, is given by a set map $F : \operatorname{Obj}\mathfrak{C} \to \operatorname{Obj}\mathfrak{D}$ together a morphism $F_{A,B} : \mathfrak{C}(A,B) \to \mathfrak{D}(FA,FB)$ in $\mathfrak{V}$ for each $A, B \in \operatorname{Obj}\mathfrak{C}$, such that the following diagrams commute.

$$\begin{array}{ccc} \mathfrak{C}(B,C)\otimes\mathfrak{C}(A,B) & \xrightarrow{\circ_{A,B,C}} & \mathfrak{C}(A,C) \\ {\scriptstyle F_{B,C}\otimes F_{A,B}}\downarrow & & \downarrow{\scriptstyle F_{A,C}} \\ \mathfrak{D}(FB,FC)\otimes\mathfrak{D}(FA,FB) & \xrightarrow[\circ_{FA,FB,FC}]{} & \mathfrak{D}(FA,FC) \end{array}$$

$$\begin{array}{ccccc} & & I & & \\ & {\scriptstyle j_A}\swarrow & & \searrow{\scriptstyle j_{FA}} & \\ \mathfrak{C}(A,A) & & \xrightarrow{F_{A,A}} & & \mathfrak{D}(FA,FA) \end{array}$$

The above diagrams simply serve to indicate that a $\mathfrak{V}$-category functor must respect identity and composition, as we would have it in the standard case. Of course, setting $\mathfrak{V} = $ Set, we recover the standard definition of a functor.

Now that we have functors between $\mathfrak{V}$-categories, we may be tempted to contrive the definition of natural transformations between such categories. In the case of $\mathfrak{V}$=Set, we understand a natural transform $\eta : F \to G$ between functors $F, G : \mathfrak{C} \rightrightarrows \mathfrak{D}$ to be a collection of arrows $\eta_A : FA \to GA$ for each object $A \in \mathrm{Obj}\,\mathfrak{C}$. However, in the enriched context we cannot directly speak of individual arrows. As such, we employ the 'trick' of instead giving an arrow $\eta_A : I \to \mathfrak{D}(FA, GA)$ and specifying its properties so that it serves as though it were 'choosing' the correct morphism, were a map.

Def. (III) 1.0.6. Given two $\mathfrak{V}$-category functors $F, G : \mathfrak{C} \rightrightarrows \mathfrak{D}$, a $\mathfrak{V}$-natural transform $\eta : F \to G$ is given by a family of arrows $\eta_A : I \to \mathfrak{D}(FA, GA)$ indexed by $\mathrm{Obj}\,\mathfrak{C}$ such the the following diagram commutes, for all $A, B \in \mathrm{Obj}\,\mathfrak{C}$.

$$
\begin{array}{ccccc}
I \otimes \mathfrak{C}(A,B) & \xleftarrow{\lambda^{-1}_{\mathfrak{C}(A,B)}} & \mathfrak{C}(A,B) & \xrightarrow{\rho^{-1}_{\mathfrak{C}(A,B)}} & \mathfrak{C}(A,B) \otimes I \\
\downarrow{\scriptstyle \eta_B \otimes F_{A,B}} & & & & \downarrow{\scriptstyle G_{A,B} \otimes \eta_A} \\
\mathfrak{D}(FB,GB) \otimes \mathfrak{D}(FA,FB) & & & & \mathfrak{D}(GA,GB) \otimes \mathfrak{D}(FA,GA) \\
& \searrow{\scriptstyle \circ_{FA,FB,GB}} & & \swarrow{\scriptstyle \circ_{FA,GA,GB}} & \\
& & \mathfrak{D}(FA,GB) & &
\end{array}
$$

If $\mathfrak{V}$ = Set, then we recover the standard definition of a natural transform, viz., that it must commute with functorial images of arrows. To see this, recall that in Set, $I = \{\star\}$, and so to give η_A is to give a single arrow $FA \to GA$. Then, if we trace out the commutative diagram, beginning with an $f \in \mathfrak{C}(A,B)$, we find the requirement $\eta_B Ff = Gf\eta_A$ – precisely the familiar naturality square.

There is much that can be said for the theory of enriched categories – for example, we may attempt to recast all of the results of the standard theory in the enriched setting. By and large, this has been done (enriched adjunctions, limits, Yoneda, *etc.*) and the results have had a profound influence on the direction of the theory and formulation of the "higher category" theory. For lack of time and direct applicability to later sections, the author regrets that such directions have not been included in this work.

In what follows, we will examine some select examples of enrichment which will play a central role in later chapters.

2. Semi-additive categories

Now that the defining facets of enriched categories have been made clear, we turn our attention to a particular case of enrichment, viz., categories enriched over $(\text{Set}_\bullet, \wedge)$ where $\wedge$ is the smash product of spaces[1].

Should we work carefully through the diagrams in def. (III) 1.0.1, we see that a category enriched over $\text{Set}_\bullet$ has a distinguished element in each morphism object, and that composition takes distinguished elements to distinguished elements. Due to our algebraic inclinations, we say that such distinguished morphisms are called zero morphisms, and abstracting, we reach the following definition.

Def. (III) 2.0.1. A category has zero morphisms if $(\forall A, B \in \operatorname{Obj}\mathfrak{C})\exists 0_{AB} \in \mathfrak{C}(A, B)$ such that the following diagram commutes for all $A, B, C \in \operatorname{Obj}\mathfrak{C}$ and all $f : A \to B$ and $g : B \to C$.

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ {\scriptstyle 0_{AB}}\downarrow & \searrow^{0_{AC}} & \downarrow{\scriptstyle 0_{BC}} \\ B & \xrightarrow[\;g\;]{} & C \end{array}$$

That is, there is a system of morphisms which are biconstant in a compatible way.

Prop. (III) 2.0.2. *Zero morphism systems are unique if they exist.*

Proof. Let 0 and $0'$ be two systems of zero morphisms over a category. Consider that for all objects A, B, C we must have $0_{A,C} = 0'_{B,C} 0_{A,B} = 0'_{A,C}$. ■

It may be observed that every category with zero morphisms can be seen as enriched over $(\text{Set}_\bullet, \wedge)$, including specifically $\text{Set}_\bullet$. Although such an enrichment is a structural property, we may reach it through an entirely different avenue.

Def. (III) 2.0.3. The zero object of a category, should it exist, is an object that is both initial and terminal.

Prop. (III) 2.0.4. *A category with a zero object has zero morphisms.*

Proof. The proof is trivial as every arrow factors through the zero object, and universal properties necessitate the rest. ■

Thus, the presence of a particular *object* in a category can determine a *structural* property. Moreover, we have a partial converse in the presence of specific morphisms.

Prop. (III) 2.0.5. *In a category with zero morphisms, the following are equivalent:*

1. *There is a zero object*
2. *There is a terminal object*
3. *There is an initial object.*

[1] Recall that smash product is defined as $(A, a_0) \wedge (B, b_0) = A \times B / \sim$ where $(a, b_0) \sim (a_0, b)$.

Proof. It is clear that (1) implies (2) and (3). We show only that (2) implies (1), the other implication follows by dualisation.

Let 1 be the terminal object. That for every object C in the category there exists a morphism $1 \to C$ is clear by the existence of zero morphisms. We need only show that $0_{1,C}$ is unique. To that end, let $f : 1 \to C$ be an arrow in the category. Recall that $\mathfrak{C}(1,1) = \{\mathrm{id}_1\}$ and so $f = f\,\mathrm{id}_1 = f\,0_{1,1} = 0_{1,C}$, thus 1 is initial. ■

In order to drive home the point that there is no full converse to prop. (III) 2.0.4,

Non-example (III) 2.0.6

> Consider a ring as a monoid under multiplication and view it as a one-object category. This category has a zero morphism, but no initial or terminal objects.

Remark (III) 2.0.7. In homage to its enriched heritage, we say that a category enriched over $\mathrm{Set}_\bullet$ which has a zero object is a *pointed category.* ◁

A remarkable property of categories with zero morphisms (and so of pointed categories) is the existence of a very special morphism from the coproduct of a collection of objects, to the product of that same collection, when both exist. In order to enable effective discussion of this, we make the following small definition.

Def. (III) 2.0.8. In a category with zero morphisms, for any pair of objects A, B, define $\delta_{A,A} = \mathrm{id}_A$ and $\delta_{A,B} = 0_{A,B}$ when $A \neq B$. When a collection of objects $(C_i)_{i\in I}$ is considered, we write $\delta_{j,k}$ for δ_{C_j,C_k}.

Prop. (III) 2.0.9. *In a category with zero morphisms, if the collection of objects $(C_i)_{i\in I}$ has both a product and a coproduct, then there exists a unique morphism $\alpha : \coprod C_i \to \prod C_i$ such that $\pi_k \alpha \iota_i = \delta_{i,k}$.*

Proof. For each $k \in I$ we have a unique arrow $[(\delta_{ik})_{i\in I}] : \coprod C_i \to C_k$ such that $[(\delta_{ik})_{i\in I}]\iota_j = \delta_{jk}$. With these arrows we define $\alpha = \left\langle ([(\delta_{ij})_{i\in I}])_{j\in I} \right\rangle : \coprod C_i \to \prod C_i$ to be the unique arrow with projections $\pi_k \alpha = [(\delta_{ik})_{i\in I}]$. Uniqueness follows easily by universal property. ■

The observant reader will here notice

Cor. (III) 2.0.10. *In a category with zero morphisms, if the collection of objects $(C_i)_{i\in I}$ has both a product and a coproduct, then*

$$\left\langle \left(\left[(\delta_{ij})_{i\in I} \right] \right)_{j\in I} \right\rangle = \left[\left(\left\langle (\delta_{ij})_{j\in I} \right\rangle \right)_{i\in I} \right] : \coprod C_i \to \prod C_i$$

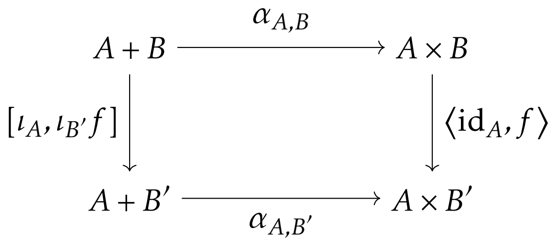
Later we shall see a sense in which this statement is obviously true, but for the time being we allow the further exploration of the properties (desired and inherent) of α to guide us onward.

A first inroad into the properties of α may be that of asking how it 'transforms' as the underlying components of the (co)product change under morphisms. More directly, we may wish to know whether, for binary (co)products, α is natural.

Prop. (III) 2.0.11. *In a category with zero morphisms, if all pairs of objects admit a product and a coproduct, then* $\alpha_{A,B} : A + B \to A \times B$ *as defined in prop. (III) 2.0.9 is a natural transformation between the bifunctors* $\times, + : \mathfrak{C} \times \mathfrak{C} \rightrightarrows \mathfrak{C}$.

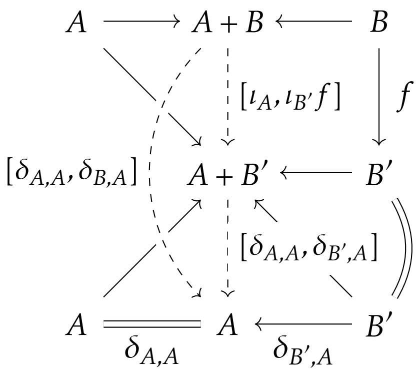
Proof. We prove only naturality in the second argument explicitly here, as naturality in the first is entirely similar. Then, a natural transform that is independently natural in both arguments is binatural and the proof is completed.

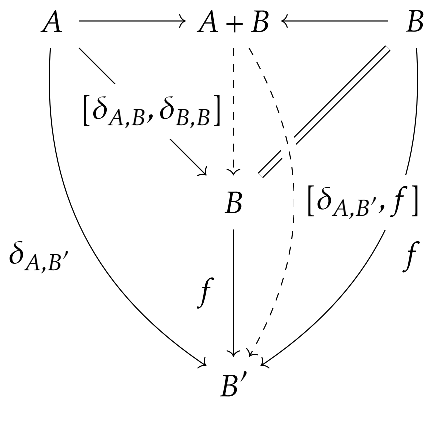
To this end, fix objects A, B and arrow $f : B \to B'$ in the category, we desire that the following diagram commute.



Given that we have $g = [\delta_{A,A}, \delta_{B,A}] : A + B \to A$ and $h = [\delta_{A,B'}, f] : A + B \to B'$ we must have a unique arrow $u : A + B \to A \times B'$ such that $\pi_A u = g$ and $\pi_{B'} u = h$. However, we have two potential candidates, viz., $\langle \mathrm{id}_A, f \rangle \alpha_{A,B}$ and $\alpha_{A,B'}[\iota_A, \iota_{B'} f]$. To ensure that they are both candidates, we must check that they satisfy the above-mentioned identities. This is immediate in all cases, but to elucidate matters, we expand $\pi_A \alpha_{A,B'}[\iota_A, \iota_{B'} f]$, $\pi_A \langle \mathrm{id}_A, f \rangle \alpha_{A,B}$ and $\pi_B \langle \mathrm{id}_A, f \rangle \alpha_{A,B}$.

For the first, by prop. (III) 2.0.9 we have $\pi_A \alpha_{A,B'}[\iota_A, \iota_{B'} f] = [\delta_{A,A}, \delta_{B',A}][\iota_A, \iota_{B'} f]$ but the right-hand side is equal to $[\delta_{A,A}, \delta_{B,A}] = g$ as the diagram below left commutes (the canonical inclusion maps are unlabelled, and note that $\delta_{B'A} f = \delta_{B,A}$).





That $f[\delta_{A,B}, \delta_{B,B}] = [\delta_{A,B'}, f] = h$ follows from the above-right commuting diagram and so gives us $\pi_B \langle \mathrm{id}_A, f \rangle \alpha_{A,B} = \pi_B \langle [\delta_{A,A}, \delta_{B,A}], f[\delta_{A,B}, \delta_{B,B}] \rangle = f[\delta_{A,B}, \delta_{B,B}] = h$. It is a comparatively simple matter to see that

$$\pi_A \langle \mathrm{id}_A, f \rangle \alpha_{A,B} = \pi_A \langle [\delta_{A,A}, \delta_{B,A}], f[\delta_{A,B}, \delta_{B,B}] \rangle = [\delta_{A,A}, \delta_{B,A}] = g$$

That the final identity, $\pi_B \alpha_{A,B'}[\iota_A, \iota_{B'} f] = h$, holds can be seen from a universal property argument entirely similar to the one given in the above-left diagram. ■

We can, of course, require even more of our special morphism.

Def. (III) 2.0.12. In a category with zero morphisms, if the unique arrow given in prop. (III) 2.0.9, $\alpha : \coprod C_i \to \prod C_i$, is an ismorphism then we say that $(C_i)_{i\in I}$ admit a biproduct, and write $\bigoplus C_i$ for its product and coproduct.

Remark (III) 2.0.13. There is a subtlety here which bears expanding. If α is indeed an isomorphism, then we understand $\prod C_i \cong \coprod C_i$, but only $\prod C_i$ is equipped with projections π_i and only $\coprod C_i$ is equipped with inclusions ι_i. To this end, when we write $(\bigoplus C_i, \pi_i, \iota_i)$ we understand there to be some slight of hand, as it is not the case that the domain of π_i is *equal* to the codomain of ι_i – somewhere, we must account for α. It is a simple matter to see that if we define $\pi_i : \bigoplus C_j \to C_i$ as π_i of $\prod C_j$ and $\iota_i : C_i \to \bigoplus C_j$ as $\alpha\iota_i$ where the ι_i are from $\coprod C_i$ then we have $\pi_i\iota_j = \delta_{ij}$. Importantly, if we were to define matters the other way around, the equality would still hold. Thus, in some sense, which of $\prod C_i$ and $\coprod C_i$ we set to be equal to $\bigoplus C_i$ does not change the relationship that the inclusions and projections of the biproduct have. ◁

Def. (III) 2.0.14. In a category with zero morphisms, if all finite collections of objects admit biproducts then the category is called semi-additive.

Example (III) 2.0.15

> It is easy to see that finite products and coproducts of commutative groups (and monoids) coincide, and that Ab and CMon both have zero morphisms and have all finite biproducts.

Remark (III) 2.0.16. The reason here that we choose to require the existence of only finite (as opposed to arbitrary) biproducts is so that we restrict ourselves to a reasonable generalisation of 'algebraic' categories (examples above). In particular, semi-additive categories will later lead to additive categories and later still will inspire abelian categories whose very design, in so far as we are concerned, is inspired by the desire to support homological theories in a unified manner. ◁

In particular then, all semi-additive categories are pointed. We know that we may also restate the above as the category admitting binary biproducts and having an initial (equivalently terminal) object. With these two definitions, we are ready to prove yet another interesting case of the presence of particular objects providing a global structure.

Prop. (III) 2.0.17. *Every semi-additive category is canonically enriched over* CMon, *the category of commutative algebraic monoids with the canonical cartesian monoidal structure.*

Proof (Sketch). In order to prove this, we need to demonstrate that every set of morphisms admits an algebraic commutative monoid structure which is preserved by composition.

To begin then, recall the results of prop. (II) 1.3.7 and its dual statement. Then, let $\mathfrak{C}$ be semi-additive and fix two objects in the category, A and B, and consider $\mathfrak{C}(A,B)$. We wish to define addition, so take $f, g \in \mathfrak{C}(A,B)$ and define $f+g \in \mathfrak{C}(A,B)$ to be the composite

$$A \xrightarrow{\Delta} A \oplus A \xrightarrow{\quad f\oplus g \quad} B \oplus B \xrightarrow{\nabla} B$$

The commutativity of addition immediately follows from the commutativity of either the comonoid structure on A or monoid structure on B. The associativity is a trivial consequence of the fact that the monoidal structures in question have an associator, coupled with a standard universal property argument. That addition respects $0_{A,B}$ is again the result of a universal property argument involving λ and ρ.

Finally, we must show that composition is a monoid homomorphism from the biproduct. We already know that it preserves $0_{A,B}$ by definition, so we must only check that $(f+g)h = fh + fg$ and $h(f+g) = hf + hg$, but again these follow from universal property arguments based on ∇ and Δ respectively. The curious reader is encouraged to explore the diagrams, but we will not belabour the proof here. ■

Remark (III) 2.0.18. There is one lamentable aspect of this otherwise glorious result. Due to the cartesian monoidal structure of CMon, in a category simply enriched over CMon (and not semi-additive) composition is not required to be bilinear (over $\mathbb{N}$) in the sense that we do not automatically inherit zero morphisms from such an enrichment. This will not be a problem with Ab later, but adds an extra factor to consider here. ◃

Now that we have established addition of morphisms canonically, we will attempt to add a variety of morphisms – especially those relating to biproducts.

Prop. (III) 2.0.19. *In a semi-additive category, if the finite collection C_i admits a biproduct $C = \bigoplus C_i$, then $\sum \iota_i \pi_i = \mathrm{id}_C$.*

Proof. The proof is a straightforward universal property argument. ■

Before we proceed with a generalisation of the above argument, we consider here the nature of the commutative monoids themselves.

Prop. (III) 2.0.20. *In the canonical enrichment of a semi-additive category over* CMon, *if additionally $\iota_k \pi_j = \delta_{jk}$ then morphism monoids are cancellative.*

Proof. In such a category $\mathfrak{C}$ we wish to show that $a + b = a + c \implies b = c$ for all $a, b, c \in \mathfrak{C}(A, B)$ for any objects $A, B \in \mathrm{Obj}\,\mathfrak{C}$. To do so, we recall that we defined $a + b = \nabla(a \oplus b)\Delta$ and leverage universal properties to our advantage.

Observe that $\Delta = \iota_0 + \iota_1$ by universal property of biproduct. Thus, $\Delta(\pi_0 + \pi_1) = \iota_0\pi_0 + \iota_1\pi_1 + \iota_0\pi_1 + \iota_1\pi_0 = \mathrm{id} + 0$ by assumption and prop. (III) 2.0.19. Ergo by precomposition with $\pi_0 + \pi_1$, that $\nabla(a \oplus b)\Delta = \nabla(a \oplus c)\Delta$ implies $\nabla(a \oplus b) = \nabla(a \oplus c)$. A universal property argument shows that $\nabla(a \oplus c)\iota_1 = c$ so that $\nabla(a \oplus b) = \nabla(a \oplus c)$ implies $b = c$ by precomposition with ι_1. ■

Now we return to the context of prop. (III) 2.0.19 and show a small, perhaps trivial, but nevertheless consequential generalisation that states that an arrow between biproducts is completely determined by its actions on components.

Prop. (III) 2.0.21. *In a semi-additive category, a morphism $f : \bigoplus_1^n A_i \to \bigoplus_1^m B_i$ is uniquely determined by the nm-many morphisms ${f^i}_j = \pi'_i f \iota_j$, where $(\bigoplus_1^n A_i, \pi', \iota')$ and $(\bigoplus_1^m B_i, \pi, \iota)$ are finite biproducts.*

Proof. This is a consequence of universal properties – specifying the collection $\pi_i f$ determines f uniquely into $\bigoplus B_i$ from its projection onto each B_i. Then, specifying ${f^i}_j = \pi_i f \iota_j$ for fixed i and determines $\pi_i f$ uniquely from $\bigoplus A_i$ from $\pi_i f \iota_j$ on A_j. ■

This result suggests of itself something with which we are very familiar. Indeed, the notation was chosen so as to all but prove the following.

Prop. (III) 2.0.22. *In a semi-additive category, let $A = \bigoplus A_i$, $B = \bigoplus B_i$, and $C = \bigoplus C_i$ be finite biproducts with arrows $f, g : A \rightrightarrows B$ and $h : B \to C$. Then $(hf)^i{}_j = \sum_k h^i{}_k f^k{}_j$ and $(g+f)^i{}_j = g^i{}_j + f^i{}_j$.*

Proof. Let $(A, \pi'', \iota''), (B, \pi', \iota'), (C, \pi, \iota)$ be the finite biproducts in question. To see the first result, consider that

$$\sum_k h^i{}_k f^k{}_j = \sum_k \pi''_i h \iota'_k \pi'_k f \iota_j = \pi''_i h \left(\sum_k \iota'_k \pi'_k \right) f \iota_j = \pi''_i h \,\mathrm{id}_B f \iota_j = (hf)^i{}_j$$

where the penultimate equality is due to prop. (III) 2.0.19, and the antepenultimate one is due to prop. (III) 2.0.17. The second result is universal property argument coupled with the distributivity of composition. ■

We have suddenly arrived at something that the reader is reasonably expected to find surprising, should he not have encountered it before. Semi-additive categories lend to their morphisms a calculus of matrices! That is, props. (III) 2.0.21 and (III) 2.0.22 combine to allow us to specify morphisms involving finite biproducts as matrices, where we have extended the notation in the obvious manner as indicated below on the left- and right- most arrows, and where composites correspond to matrix products and parallel sums to matrix sums.

$$A \xrightarrow{\begin{pmatrix} f_1 & \cdots & f_n \end{pmatrix}} \bigoplus^n B_i \xrightarrow{\begin{pmatrix} g^1{}_1 & \cdots & g^1{}_m \\ \vdots & \ddots & \vdots \\ g^n{}_1 & \cdots & g^n{}_m \end{pmatrix}} \bigoplus^m C_i \xrightarrow{\begin{pmatrix} h^1 \\ \vdots \\ h^m \end{pmatrix}} D$$

With this new understanding we return to cor. (III) 2.0.10 and observe that it is simply the statement that specifying the contents of a matrix in row-major or column-major order does not change the matrix *en masse*. Moreover, by omitting a few subscripts, we can recast prop. (III) 2.0.17 in matrix terms to discover that it essentially showed the 'obvious' statement $f + g = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Further still, prop. (III) 2.0.19 is the general version of the statement that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

This is, of course, extremely exciting and interesting and the topic would appear to be exploding with questions, the most obvious of which is perhaps: "FinVect$_{\mathbb{F}}$ has biproducts, and FinVect$_{\mathbb{F}}$ is certainly enriched over CMon – do we recover standard matrix linear algebra in this fashion?" and "is there a reasonable generalisation of conjugate transpose?"The answer in both cases is, astoundingly, yes! Regrettably, however, we shall not explore such avenues as they would lead us far astray.

We have seen then, how requiring the existence of all finite biproducts gives us, chiefly, a canonical enrichment over CMon. Conversely, if we begin with an enrichment over CMon, we may define biproducts as follows.

Def. (III) 2.0.23. The biproduct of objects A_0, A_1 is an object B equipped with morphisms $\pi_i : B \to A_i$ and $\iota_i : A_i \to B$ such that $\pi_i \iota_j = \delta_{ij}$ and $\sum \iota_i \pi_i = \mathrm{id}_B$.

This definition is entirely compatible with our earlier definition when the two are applicable, such as in the case of the canonical enrichment of semi-additive categories. Thus, it is no surprise that

Prop. (III) 2.0.24. *In a* CMon-*enriched category with zero morphisms, the existence of the following are equivalent:*

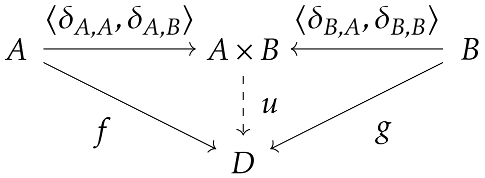
1. *All finite products*
2. *All finite coproducts*
3. *All finite biproducts*

Proof. We begin with (3) implying (1) and (2). The nullary case is obviously true, and we will show only that (3) implies (1), as the the other result is achieved by dualisation. Thus, assume that the biproduct B of the finite collection $(A_i)_{i \in I}$ exists. Given maps $f_i : C \to A_i$ we may form $u = \sum \iota_i f_i$ as an arrow $u : C \to B$ with $\pi_i u = f_i$. To see that u is unique, suppose there was an arrow $v : C \to B$ with $\pi_i v = f_i$, then $v = \mathrm{id}_B v = (\sum \iota_i \pi_i) v = (\sum \iota_i f_i) = u$.

Next we show that (1) implies (3), the other result follows by duality. Moreover, our proof will be for the two object case, as this implies the finite case as we have already the nullary case (prop. (III) 2.0.5).

Fix objects A and B in the category, and consider that we have two canonical endomorphisms of $A \times B$, viz., $\langle \delta_{A,A}, \delta_{B,A} \rangle \pi_A$ and $\langle \delta_{B,A}, \delta_{B,B} \rangle \pi_B$. As such, we may form the sum $\sigma = \langle \delta_{A,A}, \delta_{B,A} \rangle \pi_A + \langle \delta_{B,A}, \delta_{B,B} \rangle \pi_B$ and observe that $\pi_A \sigma = \pi_A$ and $\pi_B \sigma = \pi_B$ allowing us to conclude that $\sigma = \mathrm{id}_{A \times B}$ – this will be crucial.

Now suppose that we have an object D with morphisms $f : A \to D$ and $g : B \to D$. We wish to find a unique morphism u such that the following diagram commutes, making $A \times B$ isomorphic to $A + B$



There certainly *exists* a u with the required properties, $u = f\pi_A + g\pi_B$. For uniqueness, suppose there was a $v : A \times B \to D$ and consider that $v\sigma = u\sigma$ by the required properties of v, but $\sigma = \mathrm{id}_{A \times B}$ so $v = u$. ■

As a last result for this section, not only is the above-define biproduct thus identical, but it even supports the same addition of morphisms.

Prop. (III) 2.0.25. *In a* CMon-*enriched category with biproducts, the sum of the parallel arrows* $f, g : A \rightrightarrows B$ *admits the identity* $f + g = \nabla_B (f \oplus g) \Delta_A$.

Proof. Observe that $\Delta_A = \iota_0 + \iota_1$ by universal property so that $\nabla_B (f \oplus g) \Delta_A = \nabla_B (f \oplus g)(\iota_0 + \iota_1) = \nabla_B (f \oplus g) \iota_0 + \nabla_B (f \oplus g) \iota_1 = \nabla_B \iota_0 f + \nabla_B \iota_1 g = f + g$. ■

3. Strict 2-categories

To do (14)

3.1. Monads

To do Monads from 2-cat perspective (15)

3.2. Adjunctions

IV. Structures

1. Bicategories

To do Horizontal categorification of monoidal (16)

1.1. Span $\mathfrak{C}$

To do Use proofs from category object (17)

To do Monad here is internal category (18)

1.2. Mat $\mathfrak{C}$

To do (19)

2. Double categories

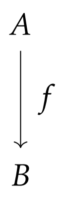
Def. (IV) 2.0.1. A double category is a category object in Cat.

Said another way, a double category is a pair of categories and parallel functors between them $\partial_0, \partial_1 : \mathfrak{A} \rightrightarrows \mathfrak{O}$ equipped with a functor $\bullet : \mathfrak{A} \times_{\mathfrak{O}} \mathfrak{A} \to \mathfrak{A}$ and a functor $I^h : \mathfrak{O} \to \mathfrak{A}$ such that this collection forms a monoid in $\mathfrak{O}$ – Precat.

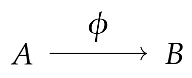
A double category is a genuinely new type of categorical structure, and so we must let go of the comfort and trappings of what we imagine a categorical structure might look like. With that said, we still have a collection of objects in our double category. In particular, for reasons that will soon become clear, we think of the objects of the object category as the *0-cells* of our double category. From the above description in terms of functors, we see that $\mathfrak{A}$-objects have ∂-domain and ∂-codomain $\mathfrak{O}$-objects and so we consider such objects *1-cells* in our double category – but they form a class of 1-cells distinct from that formed by the arrows of the object category which have only domain and codomain inherited from their ambient category, $\mathfrak{O}$. We enforce this distinction by writing the objects of $\mathfrak{A}$ as *horizontal* arrows and the arrows of $\mathfrak{O}$ as *vertical* arrows.

Pressing further, $\mathfrak{A}$-arrows have ∂-domain and ∂-codomain $\mathfrak{O}$-arrows and so we consider $\mathfrak{A}$-arrows to be *2-cells* of a sort between vertical arrows in our double category. Of course, $\mathfrak{A}$-arrows also have $\mathfrak{A}$-domain and $\mathfrak{A}$-codomain objects – horizontal arrows in our double category – and so the complete data of an $\mathfrak{A}$-arrow, a 2-cell in our double category, is given by its $\mathfrak{A}$-domain and $\mathfrak{A}$-codomain objects manifesting as horizontal arrows in our double category as well as its ∂-domain and ∂-codomain $\mathfrak{O}$-arrows, vertical arrows in our double category. However, this is not the end of the story for $\mathfrak{A}$-arrows.

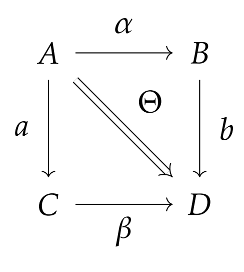
It is not the case that the vertical and horizontal double categorical domains of the 2-cell are independent or arbitrary. Writing $\Theta : \alpha \to \beta$ for an arrow in $\mathfrak{A}$, observe that it is the case that $\mathrm{dom}_{\mathfrak{O}}\, \partial_0 \Theta = \partial_0\, \mathrm{dom}_{\mathfrak{A}}\, \Theta$ by functoriality so that the domain vertical arrow of Θ and the domain horizontal arrow of Θ both begin at the same object in our double category. Similarly, $\mathrm{dom}_{\mathfrak{O}}\, \partial_1 \Theta = \partial_1\, \mathrm{dom}_{\mathfrak{A}}\, \Theta$ tells us that the codomain vertical arrow of Θ begins where the domain horizontal arrow of Θ ends. Exploring the remaining identities affords us the following picture of the constituents of a double category, where we write $A, B, \ldots$ for the objects.



(a) $f \in \mathrm{Mor}\, \mathfrak{O}$



(b) $\phi \in \mathrm{Obj}\, \mathfrak{A}$, $\partial_0 \phi = A$, $\partial_1 \phi = B$

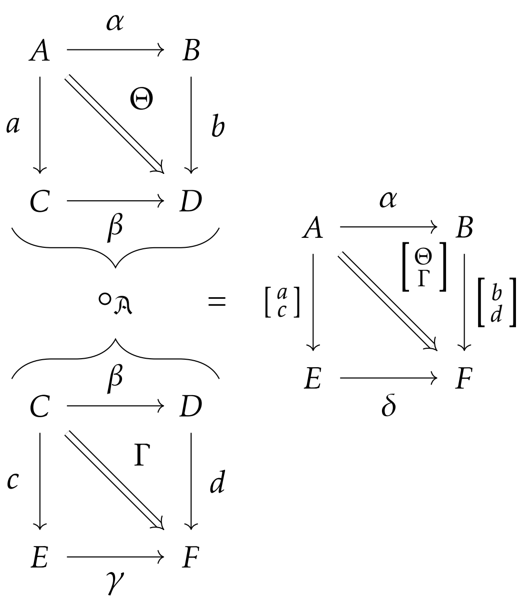


(c) $\Theta \in \mathrm{Mor}\, \mathfrak{A}$, $\mathrm{dom}\, \Theta = \alpha$, $\mathrm{cod}\, \Theta = \beta$, $\partial_0 \Theta = a$, $\partial_1 \Theta = b$

With this presentation in hand, we feel motivated to term the $\mathfrak{A}$-arrows *squares* – a term whose introduction affords a concise description of the constituents of a double category. A double category thus comprises objects, horizontal and vertical arrows and squares, arranged as above. So as to further enforce a distinction, the identity arrows of $\mathbb{O}$ are written as id^v_A, but the identity arrows of $\mathfrak{A}$ are written as I^v_α.

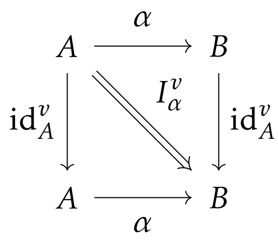
Now that we understand the contents of a double category, we may address composition. To begin, notice that there are three distinct composition operations at play: the composition of vertical arrows within $\mathbb{O}$, the composition of arrows within $\mathfrak{A}$, and the specified composition operation $\bullet$. We will treat each of these individually before finally examining how they interact.

The first is quickly dispensed with, the category $\mathbb{O}$ lives vertically within our double category. $\mathfrak{A}$-composition, on the other hand, manifests itself as a vertical composition of squares. To see this, consider that for a composable pair of $\mathfrak{A}$-arrows $\Theta : \alpha \to \beta$, $\Gamma : \beta \to \delta$ we have $\mathrm{dom}\,\Gamma = \mathrm{cod}\,\Theta$ so that the square Θ has a bottom edge identical to the top edge of the square Γ. Moreover, notice that $\partial_i(\Gamma \circ_{\mathfrak{A}} \Theta) = \partial_i\Gamma \circ_{\mathbb{O}} \partial_i\Theta$ by functoriality so that the vertical arrows forming the domain and codomain of the squares compose as expected. Pictorially, with the aid of new notation, this is expressed as follows.



It is the expectation of the author that the square bracket notation is easily understood. On vertical arrows, [] simply means composition in $\mathbb{O}$, that is, $\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$ means $c \circ_{\mathbb{O}} a$ and in the context of squares, $\left[\begin{smallmatrix} \Theta \\ \Gamma \end{smallmatrix}\right]$ means $\Gamma \circ_{\mathfrak{A}} \Theta$. While we immediately have that vertical composition of squares is associative, the idea of vertical identity squares perhaps requires some thought.

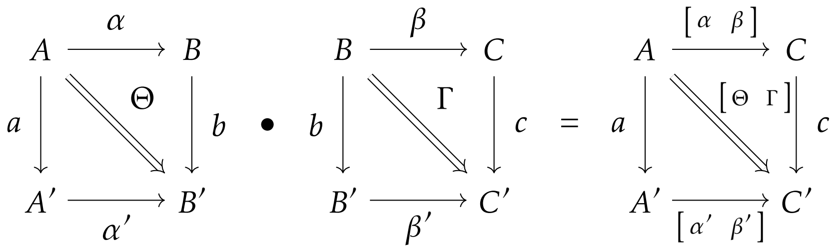
For any horizontal arrow α, we know that there is an $\mathfrak{A}$-identity arrow I^v_α and so there is a corresponding square in the double category. Considering the functorial nature of ∂_i, specifically that $\partial_i I^v_\alpha = \mathrm{id}^v_{\partial_i\alpha}$, we may deduce that these vertical identity squares have vertical edges identity and horizontal edges the horizontal arrow under consideration. It is easy to see that such squares are the identity under vertical composition.



With that established, we turn to the specified composition operation, $\bullet$. Note first that $\bullet$ is a functor and so it will have effect on both horizontal arrows and squares. In the case of horizontal arrows, the definition of the monoid from which $\bullet$ is drawn stipulates that horizontal arrows are composable when their domain and codomain objects agree in the usual manner. In a similar way, we see that squares are composable when they share a vertical edge.

$$A \xrightarrow{\alpha} B \quad \bullet \quad B \xrightarrow{\beta} C \quad = \quad A \xrightarrow{[\alpha \;\; \beta]} C$$

and



Here again we have introduced what is hopefully transparent notation, and in both instances [] stands in for $\bullet$. By the definition of a category object, we understand this horizontal composition to be appropriately associative, but again perhaps some care must be taken in considering identity horizontal arrows and squares.

Given that horizontal arrows are fundamentally objects, in order to speak of identity horizontal arrows we must turn to the functor I^h. Recall that I^h is a morphism of monoids and so is required to satisfy $\partial_i I^h = \mathrm{id}_{\mathbb{O}}$ as functors, but it is also a functor and so satisfies $\mathrm{dom}_{\mathcal{A}} I^h = I^h \mathrm{dom}_{\mathbb{O}}$. Thus, I^h bears the following interpretation, where for $I^h A \in \mathrm{Obj}\,\mathcal{A}$ we choose to write id^h_A and for $I^h a$ we write I^h_a.

In the above diagram we have made use of the identities $[\mathrm{id}^h_A \;\; \alpha] = \alpha = [\alpha \;\; \mathrm{id}^h_B]$ and $[I^h_a \;\; \Theta] = \Theta = [\Theta \;\; I^h_b]$, and we have omitted the corresponding left-hand versions. With that established, we may examine the interplay between the various forms of composition. To do so, again we recall that $\bullet$ is a functor, and so in particular it is the case that we have an interchange law

$$(\Theta \circ_{\mathcal{A}} \Lambda) \bullet (\Gamma \circ_{\mathcal{A}} \Phi) = (\Theta \bullet \Gamma) \circ_{\mathcal{A}} (\Lambda \bullet \Phi)$$

or, in our new notation, this amounts to the statement that specifying a matrix in row-major or column-major order does not alter its contents.

$$A \xrightarrow{\mathrm{id}^h_A} A \quad \bullet \quad A \xrightarrow{\alpha} B \quad = \quad A \xrightarrow{\alpha} B$$

and

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}^h_A} & A \\ a\big\downarrow & \overset{I^h_a}{\Searrow} & \big\downarrow a \\ A' & \xrightarrow[\mathrm{id}^h_A]{} & A' \end{array} \quad \bullet \quad \begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ a\big\downarrow & \overset{\Theta}{\Searrow} & \big\downarrow b \\ A' & \xrightarrow[\alpha']{} & B' \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ a\big\downarrow & \overset{\Theta}{\Searrow} & \big\downarrow b \\ A' & \xrightarrow[\alpha']{} & B' \end{array}$$

$$\begin{bmatrix}\begin{bmatrix}\Phi\\ \Gamma\end{bmatrix}\begin{bmatrix}\Lambda\\ \Theta\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}\Phi & \Lambda\end{bmatrix}\\ \begin{bmatrix}\Gamma & \Theta\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\Phi & \Lambda\\ \Gamma & \Theta\end{bmatrix}$$

Pictorially, this is the statement that the below-left squares admit unambiguous composition. In order to simplify this and future diagrams, we omit the 2-cell arrow when unambiguous or unimportant.

$$\begin{array}{cc} \begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ a\big\downarrow & \Phi & \big\downarrow b \\ A' & \xrightarrow[\alpha']{} & B' \end{array} & \begin{array}{ccc} B & \xrightarrow{\beta} & C \\ b\big\downarrow & \Lambda & \big\downarrow c \\ B' & \xrightarrow[\beta']{} & C' \end{array} \\ \begin{array}{ccc} A' & \xrightarrow{\alpha'} & B' \\ a'\big\downarrow & \Gamma & \big\downarrow b' \\ A'' & \xrightarrow[\alpha'']{} & B'' \end{array} & \begin{array}{ccc} B' & \xrightarrow{\beta'} & C' \\ b'\big\downarrow & \Theta & \big\downarrow c' \\ B'' & \xrightarrow[\beta'']{} & C'' \end{array} \end{array} \quad \to \quad \begin{array}{ccc} A & \xrightarrow{\begin{bmatrix}\alpha & \beta\end{bmatrix}} & C \\ \begin{bmatrix}a\\ a'\end{bmatrix}\big\downarrow & \begin{bmatrix}\Phi & \Lambda\\ \Gamma & \Theta\end{bmatrix} & \big\downarrow\begin{bmatrix}c\\ c'\end{bmatrix} \\ A'' & \xrightarrow[\begin{bmatrix}\alpha'' & \beta''\end{bmatrix}]{} & C'' \end{array}$$

There remain two final facets of double categories to be noted here, viz., the manner in which horizontal identity squares and vertical composition interact, and the manner in which vertical identity arrows and horizontal composition interact. By the functoriality of I^h, whenever vertical arrows a and b are composable

$$\begin{bmatrix} I^h_a \\ I^h_b \end{bmatrix} = I^h_{\begin{bmatrix} a \\ b \end{bmatrix}}$$

and whenever horizontal arrows α and β are composable,

$$\begin{bmatrix} I^v_\alpha & I^v_\beta \end{bmatrix} = I^v_{\begin{bmatrix}\alpha & \beta\end{bmatrix}}$$

We may also express these identities diagrammatically, where we have further enhanced our notation by encoding of horizontal and vertical composition through appropriate juxtaposition of squares.

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C \\ \mathrm{id}^v{}_A \Big\downarrow & I^v_\alpha & \mathrm{id}^v{}_B \downarrow & I^v_\beta & \Big\downarrow \mathrm{id}^v{}_C \\ A & \xrightarrow[\alpha]{} & B & \xrightarrow[\beta]{} & C \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{[\alpha \;\; \beta]} & C \\ \mathrm{id}^v{}_A \Big\downarrow & I^v_{[\alpha \;\; \beta]} & \Big\downarrow \mathrm{id}^v{}_C \\ A & \xrightarrow[{[\alpha \;\; \beta]}]{} & C \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}^h{}_A} & A \\ a \Big\downarrow & I^h_a & \Big\downarrow a \\ B & \xrightarrow{\mathrm{id}^h{}_B} & B \\ b \Big\downarrow & I^h_b & \Big\downarrow b \\ C & \xrightarrow[\mathrm{id}^h{}_C]{} & C \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{\mathrm{id}^h{}_A} & A \\ \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \Big\downarrow & I^h_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]} & \Big\downarrow \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \\ C & \xrightarrow[\mathrm{id}^h{}_C]{} & C \end{array}$$

Double functors and natural transforms

Def. (IV) 2.0.2. A double functor is an internal functor in Cat.

Elaborating this definition reveals that double functors send morphisms of a given type to morphisms of a matching type, and in doing so respect all units and compositions in the obvious, strict manner. Between two double functors there are two, related notions of natural transformations, viz., horizontal and vertical double natural transformations.

Def. (IV) 2.0.3. Given two parallel double functors, $F, G : \mathfrak{C} \rightrightarrows \mathfrak{D}$ a horizontal double natural transform $\tau : F \to G$ comprises

1. A $\mathfrak{C}$-object-indexed family of horizontal arrows in $\mathfrak{D}$, $\tau_A : FA \to GA$, which are natural under horizontal composition. That is, given $\alpha : A \to B$ in $\mathfrak{C}$ the equality $[\,F\alpha \;\; \tau_B\,] = [\,\tau_A \;\; G\alpha\,]$ holds.

2. A $\mathfrak{C}$-vertical-arrow-indexed family of squares in $\mathfrak{D}$,

$$\begin{array}{ccc} FA & \xrightarrow{\tau_A} & GA \\ {\scriptstyle Fa}\downarrow & \tau_a & \downarrow{\scriptstyle Ga} \\ FA' & \xrightarrow[\tau_{A'}]{} & GA' \end{array}$$

   which are natural in horizontal composition of squares, $[\,\tau_a \;\; G\Theta\,] = [\,F\Theta \;\; \tau_b\,]$. Furthermore, this family is constrained to satisfy $\tau_{\mathrm{id}^v{}_A} = \mathrm{id}^v{}_{\tau_A}$ and $\tau_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]} = \left[\begin{smallmatrix} \tau_a \\ \tau_b \end{smallmatrix}\right]$.

The definition of a vertical double natural transform may be reached by exchanging horizontal and vertical wherever they appear in the above definition.

Def. (IV) 2.0.4. Given four parallel double functors, two vertical and two horizontal double natural transformations arranged as below left, a double natural square Γ is a domain-object-indexed family of squares Γ_A, depicted below right,

$$\begin{array}{ccc} F & \xrightarrow{\tau} & G \\ {\scriptstyle f}\downarrow & \Gamma & \downarrow{\scriptstyle g} \\ F' & \xrightarrow[\sigma]{} & G' \end{array} \qquad\qquad \begin{array}{ccc} FA & \xrightarrow{\tau_A} & GA \\ {\scriptstyle f_A}\downarrow & \Gamma_A & \downarrow{\scriptstyle g_A} \\ F'A & \xrightarrow[\sigma_A]{} & G'A \end{array}$$

which are natural in the following sense of composition of squares, for Θ a square in the domain,

$$\begin{bmatrix} \Gamma_A & g_\alpha \\ \sigma_a & G'\Theta \end{bmatrix} = \begin{bmatrix} f_\alpha & \Gamma_B \\ F'\Theta & \sigma_b \end{bmatrix} = \begin{bmatrix} F\Theta & \tau_b \\ f_{\alpha'} & \Gamma_{B'} \end{bmatrix} = \begin{bmatrix} \tau_a & G\Theta \\ \Gamma_{A'} & g_{\alpha'} \end{bmatrix}$$

V. Homology

Now that we have established some of the basic definitions and elementary results concerning abelian categories, we may use this language as a platform to discuss exactness and homology functors, and ultimately to briefly phrase the classical singular homology in a more general fashion.

However, due once more to the extremely short time-frame permitted to the author, our treatment of these notions will be sparing and we shall introduce only the barest of definitions in an attempt to charge towards the statement of singular homology as economically as possible. That is to say, we shall not explore at all the elementary diagram lemmas nor indeed shall we investigate any of the theory concerning projective modules, Ext and Tor, or particular examples of chain homology beyond a simple outline of simplicial homology. Moreover, we shall omit many important statements concerning homology functors and their specific instances – statements such as the homotopy invariance of H_n which lend themselves to the greater context.

Nevertheless, we will strive to give – if only in the broadest of strokes – an idea of the foundational definitions, if not some discussion of a subset of the core objects of concern, so that further directed investigation may be made from here.

1. Kernels and co.

Continuing in the algebraic vein ushered in by the previous section, we introduce some important generalisations of essentially algebraic notions so as to provide a uniform means to discuss later concepts.

Def. (V) 1.0.1. Given $A \in \mathrm{Obj}\,\mathfrak{C}$ and $M \subseteq \mathrm{Mono}\,\mathfrak{C}$ a class of monomorphisms, an M-subobject is an isomorphism class of M-monomorphisms $m : B \to A$. That is, two M-monomorphisms $m : B \to A$ and $m' : B' \to A$ are equivalent iff there exists an isomorphism $k : B \to B'$ such that $m = m'k$. If $M = \mathrm{Mono}\,\mathfrak{C}$ then we say that $m : B \to A$ is a subobject.

Surprise (V) 1.0.2

In Set, a subobject B of a set A is the class of all injections $m : B' \to A$ such that $|B'| = |B|$, and so not any *subset* in particular. Indeed, it is certainly possible for $B' \cap A = \phi$ in general. Moreover, in Top we see that while subspaces are certainly subobjects, so is the space itself with any finer topology. Thus, subobjects do not necessarily capture the correct notion of containment that we desire when we speak of subsets, subspaces and so on. As such, we will confine future discussion to regular subojects, where $M = \mathrm{RegMono}\,\mathfrak{C}$.

Def. (V) 1.0.3. Given $A \in \operatorname{Obj}\mathfrak{C}$ and a class of epimorphisms $E \subseteq \operatorname{Epi}\mathfrak{C}$, an E-quotient object is an isomorphism class of epimorphisms $e : A \to B$. That is, two E-epimorphisms $e : A \to B$ and $e' : A \to B'$ are equivalent iff there exists an isomorphism $k : B \to B'$ such that $e = ke'$.

Surprise (V) 1.0.4

> In Mon, we find that the inclusion map $\mathbb{N} \xrightarrow{\subset} \mathbb{Z}$ is actually an epimorphism (though *not* a surjection of sets) and so $\mathbb{Z}$ is a quotient object of $\mathbb{N}$. Again then, there is a problem with simply taking all morphisms, and so we restrict attention to regular quotient objects.

With sub- and quotient- objects defined, we are tempted to generalise the standard algebraic examples of such objects.

Def. (V) 1.0.5. In a category with zero morphisms, we define the kernel of a map $f : A \to B$ to be the equaliser of f and 0_{AB}, $\ker f = \operatorname{eq}(f, 0)$, when it exists. Dually, the cokernel is given by $\operatorname{coker} f = \operatorname{coeq}(f, 0_{AB})$.

Like all equalisers, the kernel is a regular monomorphism and so bears interpretation as a regular subobject. Dually, cokernels are regular quotient objects. Moreover, as with all limits, the object itself is only unique up to isomorphism, but should we include the appropriate morphism, then the collection is unique up to unique isomorphism. Partly motivated by this, and partly by the ever-present weight of brevity, we use $\ker f$ to refer to both the object and the morphism $k : \ker f \to \operatorname{dom} f$ wherever context would disambiguate such a choice.

Non-example (V) 1.0.6

In Ring, the category of unital rings, there are no categorical kernels as there can be no zero morphisms.

Remark (V) 1.0.7. The usual subtlety about referring to limits is exacerbated in the case of (co)kernels wherein we may wish to discuss objects such as $\ker \operatorname{coker} f$. In general categories, there isn't a canonical $\operatorname{coker} f$ from which to construct $\ker \operatorname{coker} f$. As such, any proof we give for $\ker \operatorname{coker} f$ and related notions is to be carefully understood to hold for a presupposed and indeed arbitrary choice of $\operatorname{coker} f$, but not for all such objects *at once*. In particular, as long as $f \cong g$ with either a domain or a codomain isomorphism, we see that $\ker f \cong \ker g$ and $\operatorname{coker} f \cong \operatorname{coker} g$ whenever they exist. ◃

Prop. (V) 1.0.8. *In a category with zero object, for any monomorphism $m : A \to B$, $(\ker m, k : \ker m \to A) \cong (0, 0_A)$.*

Proof. For any arrow $k : K \to A$ such that $mk = 0_{AB}k$ we must have $mk = 0_{AB}k = 0_{KB} = m0_{KA}$ and so $k = 0_{KA}$. By definition 0_{KA} factors uniquely through the zero object. ∎

Cor. (V) 1.0.9. *If the category contains a zero object, then for all morphisms $f : A \to B$ with kernel and $g : C \to D$ with cokernel,* $\ker \ker f \cong 0$ *and* $\operatorname{coker} \operatorname{coker} g \cong 0$.

The reader may at this point be wondering about the reverse implication omitted from prop. (V) 1.0.8. As it turns out, it is not true in a *general* category. Indeed, we will need to first introduce the notion of abelian categories in order to satisfactorily demonstrate a sufficient condition.

Finally, we exhibit properties that we may intuitively suspect hold, based perhaps upon our experience with abelian groups.

Prop. (V) 1.0.10. *If the category contains a zero object, then the kernel of $0_{A,B} : A \to B$ is isomorphic to A.*

Proof. Observe that $0_{A,B}\mathrm{id}_A = 0_{A,B}$ and so we have a unique $u : A \to \ker 0_{A,B}$ with $ku = \mathrm{id}_A$. Then a simple universal property argument shows that $uk = \mathrm{id}_{\ker 0_{A,B}}$ so that $A \cong \ker 0_{A,B}$. ∎

Prop. (V) 1.0.11. *In a semi-additive category, if $(A = A_0 \oplus A_1, \iota, \pi)$ is a biproduct, then*

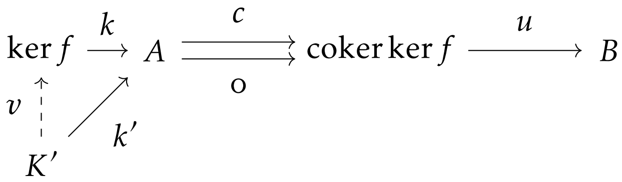
$$\iota_i \cong \ker \pi_j, \quad \pi_i \cong \operatorname{coker} \iota_j \quad (i \neq j)$$

Proof. Recall that the projections and inclusions satisfy $\pi_i\iota_j = \delta_{ij}$. We will show only that (A_i, π_i) is the coequaliser of $(0_{j,A}, \iota_j)$ for $i \neq j$ as the other proof is entirely similar.

First consider that we already have $\pi_i\iota_j = 0_{ij}$ and so we must only show that π_i is universal with respect to the coequaliser property. To that end, let $f : A \to B$ be an arrow with $f0_{j,A} = f\iota_j$. To see that there is a $u : A_i \to B$ with $u\pi_i = f$, recall that (prop. (III) 2.0.19) $f = \sum f\iota_i\pi_i$ but in this case, $f\iota_j = 0$ so that $f = f\iota_i\pi_i$ allowing us to write $u = f\iota_i$. That u is unique follows trivially from this, as if $v : A_i \to B$ had $v\pi = f$ then $v = v\pi_i\iota_i = f\iota_i = u$. ∎

Prop. (V) 1.0.12. *For arbitrary arrow f, whenever the appropriate objects exist, the following isomorphisms hold:* $\ker \operatorname{coker} \ker f \cong \ker f$ *and* $\operatorname{coker} \ker \operatorname{coker} f \cong \operatorname{coker} f$.

Proof. We show only the first as the second follows via dualisation. Let $f : A \to B$ and suppose $k : \ker f \to A$ and $c : A \to \operatorname{coker} \ker f$ exist. Observe that $fk = 0_{\ker f B} = f 0_{\ker f A}$ and so f factors as $f = cu$ for unique $u : \operatorname{coker} \ker f \to B$. Now consider the following diagram



If $ck' = 0_{A \operatorname{coker} \ker f} k'$ then $fk' = uck' = u0_{A \operatorname{coker} \ker f} k' = 0_{AB} k'$ and so we have a unique $v : \ker f \to K'$ for which $kv = k'$, and thus $\ker \operatorname{coker} \ker f \cong \ker f$. ■

Cor. (V) 1.0.13. *If every arrow has a kernel and cokernel, then $f : A \to B$ is a kernel iff.* $f \cong \ker \operatorname{coker} f$.

2. Abelian categories

Now that we have seen how enrichment of CMon is variously equivalent to the presence of specific attributes of the category, and in particular how it lends itself to the powerful notion of a biproduct, we may be tempted to exchange CMon for a category with slightly more structure and reexamine the theory.

The theory of such structured categories is both rich and deep, but the direction of the document and brevity of the allocated time period for the completion of the work have conspired to constrain discussions to topical matters. Ergo, what follows is a brief outline of some surface results and elementary definitions in this direction, the sum total of which will set the stage for discussions in the next section.

Def. (V) 2.0.1. An Ab-enriched category is a category enriched over the symmetric monoidal category Ab of abelian groups, with the tensor product as that of $\mathbb{Z}$-modules.

In order to understand what such a category represents, we must carefully examine def. (III) 1.0.1 with the knowledge that morphism objects are now abelian groups. In doing so, we see that that composition must be an abelian group homomorphism, as it is an arrow in Ab. As such, we have the curious property that composition must be bilinear with respect to the $\mathbb{Z}$-module structure of the groups and the associated tensor product. In particular then, the category has zero morphisms and we are cured of one of the ailments of CMon enrichment.

Surprise (V) 2.0.2

We already know that enriching a singleton set yields a monoid object in the underlying monoidal category. Thus, we may be led to ask, by way of considering the simplest non-trivial Ab-enriched category, what is a monoid object in Ab?

A monoid object in Ab is an abelian group G together with a multiplication morphism $\mu : G \otimes G \to G$ and an identity morphism $\eta : \mathbb{Z} \to G$ satisfying the requisite relations of unitality and associativity. Moreover, the multiplication morphism must be bilinear (it is an arrow in Ab), and thus multiplication is distributive over addition. The careful reader will be quick to note that this means that we have simply arrived at the definition of a ring!

For this reason, Ab-enriched categories are sometimes referred to as *ringoids* as they represent the 'horizontal' categorical generalisation of rings.

Joke (V) 2.0.3. A ring is a ringoid with one object.

Remark (V) 2.0.4. Observe that Abelian groups are, in particular, commutative algebraic monoids and so every Ab-enriched category is also CMon-enriched. Thus, the theory established in section 2 applies here. ◁

Example (V) 2.0.5

Ab is a closed symmetric monoidal category and so is enriched over itself, as the canonical example of an Ab-enriched category.

Remark (V) 2.0.6. Before we proceed to some results concerning Ab-enriched categories and their more structured brethren, we pause here to note that we already have a understanding of what functors between Ab-enriched categories should be. That is, we need only examine def. (III) 1.0.5 to find that such functors are morphisms of abelian groups which respect composition. ◃

Prop. (V) 2.0.7. *In an* Ab*-enriched category, for a pair of parallel arrows* $f, g : A \rightrightarrows B$*, the following conditions are equivalent and the corresponding objects are isomorphic when they exist,*

1. $\mathrm{eq}(f, g)$ *exists*
2. $\ker(f - g)$ *exists*
3. $\ker(g - f)$ *exists*

Proof. Given that $\mathrm{eq}(f, g) = \mathrm{eq}(g, f)$ it suffices to show that (1) $\iff$ (2), for example. To that end, assuming (1) where $(E, e) = \mathrm{eq}(f, g)$ we posit an arrow $h : C \to A$ such that $(f - g)h = 0_{AB}$. However, $(f - g)h = 0_{AB} \iff fg = fh$ which gives a unique arrow $u : E \to C$ by the equaliser property with $e = uh$. The reverse implication and the rest of the proof proceed simply. ■

Though this statement and its dual may be pleasing, simply enriching over Ab instead of over CMon does not bring us relevantly new, interesting results. The reader may perhaps convince himself that this is not surprising as, for example, biproducts only emerged from CMon-enrichment in the presence of finite products and a zero object. With this situation in mind, we introduce the following notion.

Def. (V) 2.0.8. An additive category is an Ab-enriched category with all finite products.

Given the contents of prop. (III) 2.0.24 we see that we may equally well have defined additive categories as Ab-enriched categories with all finite coproducts or biproducts.

If for no other reason than semantic similarity, the reader may wonder what relation additive categories have to *semi*-additive categories. Such a reader is to be congratulated for his directed questions, for they lead us to consider

Prop. (V) 2.0.9. *Any semi-additive category wherein the canonical enrichment over* CMon *extends a commutative group structure to the sets* $\mathfrak{C}(A, B)$*, is additive.*

Proof. We already know that semi-additive categories all finite biproducts, and so all we must demonstrate is that if the sets $\mathfrak{C}(A, B)$ have additive structures, then we have Ab enrichment.

This is almost completely trivial, however, as we already know that composition is distributive, associative, and unital in the proper ways (prop. (III) 2.0.17) and it is easy to see that positing the existence of additive inverses does not change any of this. Consequently, in order to prove the statement we really need only show that composition is a $\mathbb{Z}$-module morphism $\mathfrak{C}(B, C) \otimes_{\mathbb{Z}} \mathfrak{C}(A, B) \to \mathfrak{C}(A, C)$.

Thus, we aim to prove that for composable arrows $f : A \to B$, $g : B \to C$ we have $(ng)f = g(nf)$ for $n \in \mathbb{Z}$, where negative values of n are understood to have the meaning

$ng = |n|(-g)$. To this end, consider that for $n = 0$ the statement has already been proven (prop. (III) 2.0.17), and for positive n, $(ng)f = (\sum^n g)f = \sum^n gf = g(\sum^n f) = ngf$ by distributivity of composition, while the proof for negative n follows, *mutatis mutandis*. ∎

Where before we had that the biproduct structure in semi-additive categories determined a unique bimonoid structure for every object and so a canonical enrichment over CMon, in additive categories we have the following stronger and appropriately more amazing result.

Prop. (V) 2.0.10. *In an additive category, any two additive structures on the same morphism set are necessarily isomorphic.*

Proof. The proof proceeds through the following steps. We first show that for a given biproduct $A \oplus A$, $\delta = \iota_0 - \iota_1 \cong \ker \nabla_A$. That is, the difference of the inclusion maps is determined by the limit and colimit structures of the category (biproducts and equalisers), up to isomorphism. Then we show that every difference of parallel arrows admits a unique decomposition in terms of δ. Thus, $f - g$ is determined by the very same structure. Finally, we note that $f + g = f - (0 - g)$ and so the entire additive structure on the morphism sets is determined, up to isomorphism, by the limit and colimit structures of the category.

To begin then, recall that for fixed A, the unique arrow $\nabla_A : A \oplus A \to A$ is determined by the universal property $\nabla_A \iota = \mathrm{id}_A$ and, by the biproduct property we have $\nabla_A = \pi_0 + \pi_1$. Now, let $\delta = \iota_0 - \iota_1$ and observe that $\nabla_A \delta = \mathrm{id}_A - \mathrm{id}_A = 0_{A,A}$. With this, we will show that $(A, \delta) \cong \ker \nabla_A$.

$$\begin{array}{ccccc} A & \xrightarrow{\delta} & A \oplus A & \overset{\nabla_A}{\underset{0}{\rightrightarrows}} & A \\ {\scriptstyle u}\uparrow & \nearrow {\scriptstyle f} & & & \\ B & & & & \end{array}$$

We have already seen that $\nabla_A \delta = 0$, so suppose there was an arrow $f : B \to A$ with $\nabla_A f = 0$, thereby enforcing $\pi_0 f + \pi_1 f = 0$. We wish to show that there exists a unique $u : B \to A$ such that the above diagram commutes. If we let $u = \pi_0 f$ then we have $\delta u = \iota_0 \pi_0 f - \iota_1 \pi_0 f$. However, $\pi_0 f = -\pi_1 f$ by assumption so that $\delta u = \sum \iota_i \pi_i f = f$ by prop. (III) 2.0.19. Further, suppose $v : B \to A$ had $\delta v = f$. Then $\iota_0 v = f + \iota_1 v$ and so $\pi_0 \iota_0 v = \pi_0 f + \iota_1 v$, ergo $v = \pi_0 f = u$.

Now for arbitrary parallel arrows $f, g : A \rightrightarrows B$, the biproduct structure on A allows us to give a unique arrow $[f,g] : A \oplus A \to B$ satisfying the universal properties $[f,g]\iota_0 = f$ and $[f,g]\iota_1 = g$. As such, it is simple to see that $f - g = [f,g]\delta$ and so the difference of parallel arrows is determined by δ. ∎

Cor. (V) 2.0.11. *Let $\mathfrak{C}$ be an additive category, then by prop. (III) 2.0.17 $\mathfrak{C}$ is canonically enriched over* CMon*. If all the morphism sets additionally are commutative groups, then the additive structure is isomorphic to the original additive structure.*

With an elementary understanding of additive categories achieved, we briefly mention here functors between additive categories.

Def. (V) 2.0.12. A functor between additive categories is termed additive when it is an abelian group homomorphism on each morphism collection.

Happily, we have that additive functors automatically respect biproducts.

Prop. (V) 2.0.13. *A functor between additive categories is additive iff. it preserves finite biproducts.*

Proof. Recall that a biproduct (def. (III) 2.0.23) was given determined entirely by its projections, inclusions and the equations relating them. In particular, $\pi_i \iota_j = \delta_{ij}$ and $\sum \iota_i \pi_i = \mathrm{id}$. Given that each equation is preserved by an additive functor, so too are biproducts.

Conversely, suppose $F : \mathfrak{C} \to \mathfrak{D}$ preserves biproducts and consider parallel arrows $f, g : A \rightrightarrows B$. We will aim to show the middle equality in the following, thereby proving the result using prop. (III) 2.0.25.

$$F(f+g) = F(\nabla_B (f \oplus g) \Delta_A) = \nabla_{FB}(Ff \oplus Fg)\Delta_{FA} = Ff + Fg$$

We have isomorphisms $\alpha : F(A \oplus A) \to FA \oplus FA$ and $\beta : F(B \oplus B) \to FB \oplus FB$ which satisfy the properties $\pi_{FA_0}\alpha = F\pi_{A_0}$, $\alpha^{-1}\iota_{FA_0} = F\iota_{A_0}$, *etc.*, by definition. In particular then,

$$\pi_{FB_0}(Ff \oplus Fg) = Ff\pi_{FA_0} = FfF\pi_{A_0}\alpha^{-1} = F\pi_{B_0}F(f \oplus g)\alpha^{-1} = \pi_{FB_0}\beta F(f \oplus g)\alpha^{-1}$$

and similarly for the other projection, π_{FB_1}. Thus, $F(f \oplus g) = \beta^{-1}(Ff \oplus Fg)\alpha$ by universal property. Furthermore, $\pi_{FA_0}\Delta_{FA} = \mathrm{id}_{FA} = F\pi_{A_0}F\Delta_A = \pi_{FA_0}\alpha F\Delta_A$ and so by universal property, $F\Delta_A = \alpha^{-1}\Delta_{FA}$. Dually, $F\nabla_B = \nabla_{FB}\beta$ and the result follows. ■

Cor. (V) 2.0.14. *A functor is additive iff. it preserves finite biproducts iff. it preserves finite products iff. it preserves finite coproducts.*

Proof. We already have that a functor is additive iff. it preserves finite biproducts and so it remains to be shown that preserving finite biproducts is equivalent to preserving finite products (and by duality, finite coproducts). However, due to prop. (III) 2.0.24, this follows trivially. ■

Now that we are satisfied with some of the basic matter concerning semi-additive and additive categories, it is time to introduce yet more structure.

Def. (V) 2.0.15. An additive category is said to be pre-abelian if every arrow has a kernel and cokernel.

Non-example (V) 2.0.16

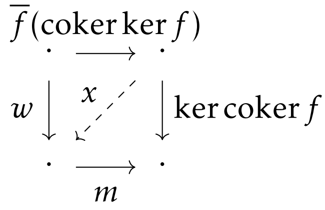
> The category of free abelian groups is not pre-abelian as there are no cokernels in general.

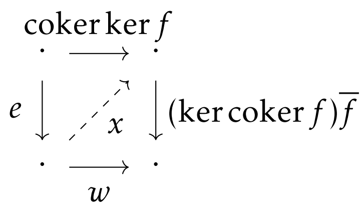
We are quick to note that pre-abelian categories are, in particular, finitely complete and cocomplete (as they have all (co)equalisers via prop. (V) 2.0.7 and (co)products by definition) and may have, in fact, been equivalently defined as Ab-enriched categories which are finitely complete and cocomplete.

Prop. (V) 2.0.17. *In a pre-abelian category,*

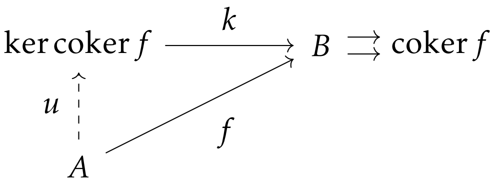
1. *Every arrow admits a canonical factorisation as* $f = (\ker \operatorname{coker} f)\overline{f}(\operatorname{coker} \ker f)$

2. *If* $f = mw$ *where* m *is a kernel, then there is a unique monomorphism* x *such that the below-left diagram commutes. Dually, if* $f = we$ *where* e *is a cokernel then there is a unique epimorphism* x *such that the below-right diagram commutes*



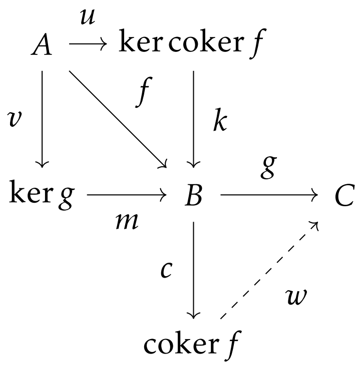


Proof. First we write $f = (\ker \operatorname{coker} f)u$ through the universal property of $\ker \operatorname{coker} f$,



Note that $f(\ker f) = 0$, but $f = (\ker \operatorname{coker} f)u$ with $\ker \operatorname{coker} f$ monic so $u(\ker f) = 0$ and we may factor $u = \overline{f} \operatorname{coker} \ker f$ by the coequaliser property.

For (2), suppose that $f = mv$ where $m = \ker g$ for some arrows $v : A \to \ker g$, $m : \ker g \to B$, and $g : B \to C$ and consider the below diagram.



By assumption the top square commutes and we have $gm = 0$. As such, $gf = gmv = 0$ and so by the cokernel property of $c = \operatorname{coker} f$ we have a unique $w : \operatorname{coker} f \to C$ for which $g = wc$. Observe then that $gk = wck = w0 = 0$ as $k = \ker c$ and so by the kernel property of $m = \ker g$ we must have a unique $x : \ker \operatorname{coker} f \to \ker g$ with $k = mx$ (which is easily seen to make x a monomorphism). Furthermore, $mv = f = ku = mxu$ and as m is a monomorphism by assumption we have $v = xu$. Dualisation of this argument completes the proof. ■

Regrettably, $\overline{f}$ is not monic or epic in general pre-abelian categories. In the same way that examining the restriction that the canonical morphism from coproducts to products be an isomorphism led to categories with interesting structure, we desire conditions for $\overline{f}$ to be an isomorphism.

Prop. (V) 2.0.18. *In a pre-abelian category, if every monomorphism is a kernel and every epimorphism is a cokernel, then for every arrow f, then the canonical arrow arising from the limit-colimit structure $\overline{f} : \operatorname{coker}\ker f \to \ker\operatorname{coker} f$ is an isomorphism.*

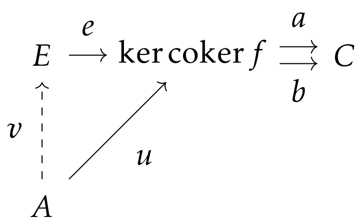
Before we prove this, we first give a minor technical result.

Lem. (V) 2.0.19. *In a pre-abelian category, if every monomorphism is a kernel and every epimorphism is a cokernel then any morphism which is monic and epic is an isomorphism.*

Proof. Assume f is monic and epic. As f is monic, $f = \ker g$ for some arrow g. However, $gf = 0$ gives $g = 0$ as f is an epimorphism. Thus, by prop. (V) 1.0.10, $f = \ker g \cong \mathrm{id}$. ■

Proof (prop. (V) 2.0.18). We show that $\overline{f}$ is both epic and monic, and consequently an isomorphism given the assumptions.

Let $u = \overline{f}\operatorname{coker}\ker f$ and consider a pair of parallel arrows $a, b : \ker\operatorname{coker} f \rightrightarrows C$ such that $au = bu$ and take their equaliser, as in the following diagram.



Recall that $f = ku = kev$ and so $m = ke$ is a monomorphism which has $f = mv$. By assumption, m is a kernel and so we may apply prop. (V) 2.0.17 to find a unique monomorphism x with $k = mx = kex$. Thus $ex = \mathrm{id}$ and so $ae = be \implies a = b$, and from $u = \overline{f}\operatorname{coker}\ker f$ being epic it easily follows that $\overline{f}$ is too. Similarly, we perform the dual of the above proof to $v = (\ker\operatorname{coker} f)\overline{f}$ to find that v is a monomorphism and so $\overline{f}$ is both monic and epic. Thus, $\overline{f}$ is an isomorphism (lem. (V) 2.0.19). ■

The observant reader will note that in a rather elementary manner, the above conditions are also necessary.

Prop. (V) 2.0.20. *In a pre-abelian category, every monomorphism is a kernel and every epimorphism is a cokernel iff. for every arrow the canonical arrow $\overline{f} : \operatorname{coker}\ker f \to \ker\operatorname{coker} f$ is an isomorphism.*

Proof. We have already shown the 'only if' part. Assume that m is a monomorphism and write $m = (\ker\operatorname{coker} m)\overline{m}(\operatorname{coker}\ker m)$ by prop. (V) 2.0.17, for $\overline{m}$ an isomorphism. By prop. (V) 1.0.8, $\ker m \cong 0$ and so $\operatorname{coker}\ker m \cong \operatorname{coker} 0 \cong \mathrm{id}$ (prop. (V) 1.0.10), making $m \cong \ker(\operatorname{coker} m)$ and thus a kernel. By dualisation, $e \cong \operatorname{coker}(\ker e)$ for e an epimorphism. ■

Cor. (V) 2.0.21. *In a pre-abelian category, where every monomorphism is a kernel and every epimorphism is a cokernel, $\ker f \cong 0 \iff f$ is a monomorphism.*

Proof. The 'only if' is given by prop. (V) 1.0.10 and the 'if' by the above proof. ■

With the sufficiency and necessity of the condition achieved, we may question the extent to which such a factorisation is unique. In order to answer this, we must transition to a setting where such a factorisation always exists.

Def. (V) 2.0.22. A pre-abelian category wherein every monomorphism is a kernel and every epimorphism is a cokernel is said to be abelian.

It is no accident of naming that we have chosen the adjective abelian. Indeed,

Example (V) 2.0.23

> Ab is an abelian category. It is Ab-enriched, it supports finite biproducts in the usual manner, every arrow has a kernel and cokernel (again in the usual manner), and every monomorphism is a kernel ($G \to H \rightrightarrows H/\mathrm{im}$) and every epimorphism is a cokernel.

Returning to the matter of factorisation – as it happens, not only is the canonical factorisation unique in abelian categories, but there is a far more general result which implies it.

Prop. (V) 2.0.24. *In an abelian category, for every commutative square of arrows $bf = f'a$ (below left), if we write $f = me$ for $m = (\ker \operatorname{coker} f)\overline{f}$ and $e = \operatorname{coker} \ker f$ and $f' = m'e'$ for m' monic and e' epic, then there exists a unique u such that the diagram below right commutes*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow[f']{} & B' \end{array} \qquad\qquad \begin{array}{ccccc} A & \xrightarrow{e} & \operatorname{coker}\ker f & \xrightarrow{m} & B \\ {\scriptstyle a}\downarrow & & \vdots\,{\scriptstyle u} & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow[e']{} & \cdot & \xrightarrow[m']{} & B' \end{array}$$

Proof. Let $k = \ker f$ so that $ek = 0$. Consequently, $mek = 0$ and $fk = 0$ and $bfk = 0$ and $f'ak = m'e'a = 0$ so that $e'a = 0$. As such, $e'a$ factors uniquely through $\operatorname{coker}\ker f$ as $ue = e'a$. Finally, $m'ue = f'a = bf = bme$ implies that $m'u = bm$. ■

Cor. (V) 2.0.25. *In an abelian category, a mono-epi factorisation of an arrow $f = me$ is unique up to isomorphism.*

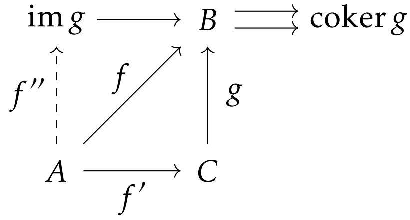
Proof. Write $f = me = m'e'$ and apply prop. (V) 2.0.24 to the degenerate square to find a unique u with $e' = ue$ and $m = m'u$ making u both epic and monic and so an isomorphism (lem. (V) 2.0.19). ■

Def. (V) 2.0.26. In an abelian category, if we write $f = (\ker \operatorname{coker} f)\overline{f}(\operatorname{coker} \ker f)$ then we say that $\operatorname{im} f = \ker \operatorname{coker} f$ and $\operatorname{coim} = \operatorname{coker} \ker f$.

Remark (V) 2.0.27. This definition is well chosen indeed. First, we have that $\operatorname{im} m \cong m$ for monomorphism m and the dual result. To see this, combine the fact that every monomorphism is a kernel with prop. (V) 1.0.12. Second, a rephrasing of prop. (V) 2.0.20 would be the statement that abelian categories are precisely the pre-abelian categories wherein the first isomorphism theorem holds ($\operatorname{im} \cong \operatorname{coim}$). ◂

Moreover, abelian categories grant a convenient notion of quotient object more readily recognisable then merely an isomorphism class of epimorphisms.

Def. (V) 2.0.28. In an abelian category, if an arrow f factors through an arrow g, then we write g/f for $\operatorname{coker}(f'' : A \to \operatorname{im} g)$ where f'' is the unique arrow arising out of the below-right commutative diagram.



The dual construction, where f and g have common domain A and $f = f'g : A \to B$, gives rise to $\ker(f'' : B \to \operatorname{coim} g)$ and we abuse notation to write $g\backslash f$ for this case.

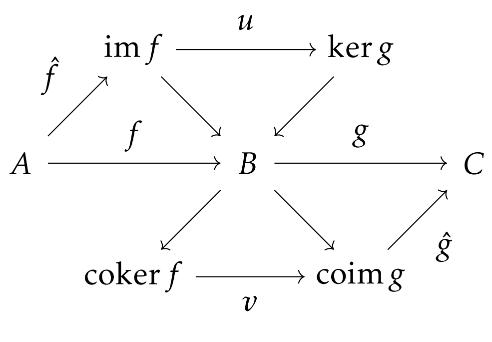
Remark (V) 2.0.29. Again, this coincides with what we might expect in Ab. The fact that f factors through g means that $\operatorname{im} f \subseteq \operatorname{im} g$ and so we can factor f through $A \to \operatorname{im} g \subseteq B$ via f'' which acts on elements as $f''(a) \mapsto f(a)$ – essentially a codomain restriction of f. Then, with the interpretation that $\operatorname{coker} f'' = \operatorname{im} g / \operatorname{im} f''$ we see that we have indeed created a reasonable definition of g/f. Moreover, if $C = B$ and $g = \mathrm{id}_C$ so that $f = f'$ then we see that $\operatorname{im} g = \mathrm{id}_C$ so that $f'' = f$ and $g/f = C/\operatorname{im} f$, exactly as we would have liked. ◁

With an eye to later sections, we consider the following statement.

Prop. (V) 2.0.30. *In an abelian category, if $gf = 0$ for composable arrows f and g, then the following are all isomorphic*

1. $\ker g / \operatorname{im} f$
2. $\operatorname{coim} g \backslash \operatorname{coker} f$
3. $\operatorname{im}(\ker g \to \operatorname{coker} f)$
4. $\operatorname{coker}(\operatorname{im} f \to \ker g)$
5. $\ker(\operatorname{coker} f \to \operatorname{coim} g)$
6. $\operatorname{coim}(\ker g \to \operatorname{coker} f)$

Proof. To begin then, let $f : A \to B$ and $g : B \to C$ be such that $gf = 0$. Using the canonical decomposition, write $f = (\operatorname{im} f)\hat{f}$ and $g = \hat{g}(\operatorname{coim} g)$ for $\hat{f}$ epic and $\hat{g}$ monic. Now, noting that $gf = 0 \implies g(\operatorname{im} f)\hat{f} = 0$ and $\hat{f}$ epic, factor $\operatorname{im} f$ through $\ker g$ and likewise $\operatorname{coker} f$ through $\operatorname{coim} g$ to arrive at the following commutative diagram.



Unlabelled, left to right, top to bottom, are $\operatorname{im} f$, $\ker g$, $\operatorname{coker} f$, and $\operatorname{coim} g$. With that achieved, consider that $\ker g / \operatorname{im} f = \operatorname{coker}(\operatorname{im} f \to \operatorname{im} \ker g)$ but $\operatorname{im} \ker g \cong \ker g$

(prop. (V) 1.0.12). Thus $\ker g/\operatorname{im} f \cong \operatorname{coker} u$ and dually $\operatorname{coim} g\backslash \operatorname{coker} f \cong \ker v$, giving (1)≅(4) and (2)≅(5). Then, writing λ for $\operatorname{coker} f \ker g$, by prop. (V) 2.0.20 we have that $\operatorname{coim}\lambda \cong \operatorname{im}\lambda$ thereby showing (3)≅(6).

Next, consider that $\lambda u = \operatorname{coker} f \ker g u = \operatorname{coker} f \operatorname{im} f = 0$ and similarly it is the case that $\operatorname{coker} f \ker g \ker\lambda = \lambda\ker\lambda = 0$ so that we may find $\ker\lambda \cong \ker(\operatorname{coker} f) = \operatorname{im} f$ with unique isomorphism $\mu : \operatorname{im} f \to \ker\lambda$ having $(\ker\lambda)\mu = u$. As such, it follows that $\operatorname{coker} u \cong \operatorname{coker}(\ker\lambda) = \operatorname{coim}\lambda$ giving (4)≅(6). Dualisation yields (3)≅(5) thereby completing the proof. ■

Remark (V) 2.0.31. In the above proof we showed that $\operatorname{im} f \cong \ker\lambda$, which is essentially the generalised version of the statement that $gf = 0$ forces $\operatorname{im} f \subseteq \ker g$, should we view matters in Ab and see λ as the composite $\ker g \subseteq B \twoheadrightarrow B/\operatorname{im} f$. ◁

To conclude this section, we note that we may have instead defined abelian categories in terms of the existence of certain objects and *derived* the additive structure on the morphism collections in a manner reminiscent to that of semi-additive categories. In particular, it is a theorem that

Thm. (V) 2.0.32. *A category is abelian iff all of the following hold,*

1. *there is a zero object,*
2. *every pair of objects has a product and a coproduct,*
3. *every arrow has a kernel and a cokernel,*
4. *every monomorphism is a kernel; every epimorphism is a cokernel*

While interesting and certainly in the spirit of the exposition so far, this proof would require a few involved technical lemmas which would consume too much spacetime. As such, the ever curious reader is directed to [Bor94] for a full and lucid exposition.

3. Exactness

Def. (V) 3.0.1. A pair of morphisms $f : A \to B$ and $g : B \to C$ are said to be exact if $\operatorname{im} f \cong \ker g$. An exact sequence in a category with zero morphisms is given by a sequence of objects (A_n) and accompanying morphisms $f_n : A_n \to A_{n+1}$ such that each pair (f_n, f_{n+1}) is exact.

Prop. (V) 3.0.2. *In an abelian category,* $\operatorname{im} f \cong \ker g \iff \operatorname{coker} f \cong \operatorname{coim} g$.

Proof. Recall that for every morphism $f = (\operatorname{im} f)\overline{f}(\operatorname{coim} f)$ and in particular $\operatorname{im} f = \ker\operatorname{coker} f$ and $\operatorname{coim} f = \operatorname{coker}\ker f$. As such, if $\ker\operatorname{coker} f \cong \ker g$ then we have $\operatorname{coker} f \cong \operatorname{coker}\ker\operatorname{coker} f \cong \operatorname{coker}\ker g = \operatorname{coim} g$ where the first isomorphism is due to prop. (V) 1.0.12. The other direction follows by dualisation. ■

Def. (V) 3.0.3. A short exact sequence is an exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

Prop. (V) 3.0.4. *In an abelian category, an exact pair of morphisms* $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$ *form a short exact sequence iff. both*

1. f *is monic,* g *is epic*

2. $f \cong \ker g$ *(equivalently,* $\operatorname{coker} f \cong g$*)*

The proof is not particularly enlightening, and relies on manipulations of the canonical decomposition of morphisms in abelian categories (props. (V) 2.0.17 and (V) 2.0.18) using a few properties of kernels and cokernels (cor. (V) 2.0.21 and props. (V) 1.0.10 and (V) 1.0.12). Nevertheless, it serves to show that we can manipulate objects in abelian categories as though they had many of the familiar properties of abelian groups.

Proof. Assume that $0 \to \cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \to 0$ forms a short exact sequence, that is, $\operatorname{im} 0 \cong \ker f$, $\operatorname{im} f \cong \ker g$ and $\operatorname{im} g \cong \ker 0$. Then, $\operatorname{im} 0 = \ker \operatorname{coker} 0 \cong \ker \operatorname{id} \cong 0$ so that $0 \cong \ker f$ and thus f is monic. Dually, g must be epic. Furthermore, recalling that $f = (\operatorname{im} f)\overline{f}(\operatorname{coim} f)$ where $\operatorname{coim} f = \operatorname{coker} \ker f \cong \operatorname{coker} 0 \cong \operatorname{id}$ we have $f \cong \operatorname{im} f$ and so $f \cong \operatorname{im} f \cong \ker g$ by exactness. Dually we may have pursued the exactness property of $\operatorname{coker} f \cong \operatorname{coim} g$ to find $\operatorname{coker} f \cong g$.

Then, assume (1) and (2) to find that $\ker f \cong 0 \cong \operatorname{im} g$, and similarly that $\operatorname{im} g = \ker \operatorname{coker} g \cong \ker 0$ so that $0 \to f$ and $g \to 0$ are exact. Moreover, $f \cong \ker g$ implies that $\ker \operatorname{coker} f \cong \ker \operatorname{coker} \ker g \cong \ker g$ and so the sequence is short and exact. ■

Prop. (V) 3.0.5. *In an abelian category,* $\cdot \xrightarrow{\ker f} \cdot \xrightarrow{\operatorname{coim} f} \cdot$ *and* $\cdot \xrightarrow{\operatorname{im} f} \cdot \xrightarrow{\operatorname{coker} f} \cdot$ *form short exact sequences for any arrow* f.

Proof. Both $\ker f$ and $\operatorname{im} f$ are monomorphisms, and both $\operatorname{coim} f$ and $\operatorname{coker} f$ are epimorphisms. As such, we must only check that $\ker f \cong \ker \operatorname{coim} f$ and $\operatorname{im} f \cong \ker \operatorname{coker} f$. The second is true by definition and the first admits a simple proof as $\ker \operatorname{im} f = \ker \operatorname{coker} \ker f \cong \ker f$ (prop. (V) 1.0.12). ■

As we can see, by assuming that the underlying category is abelian we are afforded some convenient reformulations of exactness and familiar results. Partly inspired by this, we shall restrict all further discussion in this section and those that follow, by implicitly assuming that whenever exactness or chain complexes arise the ambient category is abelian.

Given this context, and that we already know what additive functors are, we may be tempted to define 'abelian' functors which respect that key advantage of abelian categories over additive ones, viz., finite completeness and cocompleteness. To this end, we turn to finitely continuous, cocontinuous and bicontinuous functors.

Remark (V) 3.0.6. The current and popular terminology for finitely continuous, cocontinuous and bicontinuous functors (in the context of homological algebra, and somewhat beyond) is, respectively, left-exact, right-exact and exact. In an effort to remain standard in this matter we shall employ these terms. ◁

As an immediate consequence, exact functors between abelian categories preserve exact sequences and so fulfil an important role in the study of such objects. Observe further that we did not define exact functors between abelian categories to be additive, but it is a consequence of cor. (V) 2.0.14 that left- and right- exact functors between abelian categories are additive. In fact,

Prop. (V) 3.0.7. *A functor between abelian categories is left-exact iff. it is additive and it preserves kernels.*

Proof. Combine cor. (V) 2.0.14 and prop. (V) 2.0.7 and that finite completeness is equivalent to the existence of all finite products and finite equalisers. ■

This allows us to give an alternate characterisation of exact functors.

Prop. (V) 3.0.8. *An additive functor between abelian categories is exact iff. it preserves short exact sequences.*

Proof. Combine props. (V) 3.0.4 and (V) 3.0.7. ■

More than this, exact functors play a crucial role in enabling discussion of exact sequences in general abelian categories. In particular, though we have not explored "diagram chasing", many proofs are made tractable by explicitly tracing an element about a diagram as it undergoes the actions of various morphisms. Naturally, such an approach is not possible in general abelian categories and so we must find an alternative. One such is the following.

Thm. (V) 3.0.9 (Freyd-Mitchell embedding)**.** *Every small abelian category admits a fully-faithful and exact functor to* R-Mod *for some unital ring* R.

Regrettably, the proof is well beyond our means to sketch. A full version with all the necessary exposition may be found in [Fre64].

Thus, whenever we need to prove a result concerning exactness or indeed any forms of kernels or images, we may simply trace the action of maps as though we were in R-Mod and the result would be valid, independent of whether there exists an appropriate notion of elements of objects in the abelian category in question.

While this is indeed convenient, it may be troublesome that the enabling theorem is so far beyond the scope of the content thus far. [ML97] provides for us an alternate view of the scenario by defining a general notion of "members" of an object in an abelian category and showing that such members, equivalence classes of maps with codomain of interest, behave just as though they were "elements" of the object in question, thereby obviating the need for such powerful and advanced considerations as the theorem of Freyd-Mitchell.

With all of this, we would be able to consider such statements as the five lemma, given below, which prove useful in more advanced considerations in homological algebra.

Lem. (V) 3.0.10 (Five lemma)**.** *Given the below commutative diagram in an abelian category, if the top row is exact, m and p are isomorphisms, l is an epimorphism, and q is a monomorphism, then n is an isomorphism.*

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow{\scriptstyle l} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}$$

3.1. Chain homology

Def. (V) 3.1.1. In an abelian category, a chain complex is a sequence of objects labelled by integers and composable arrows between them

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

where $\partial_n \partial_{n+1} = 0$ for all n, oftentimes abbreviated as $(C_\bullet, \partial_\bullet)$ or $C_\bullet$.

We may be tempted to consider chains as objects all their own, and as such, we would require a suitable definition of morphisms between chain complexes. In what follows, we will write all maps within chains as ∂_n wherever unambiguous.

Def. (V) 3.1.2. A morphism of chains, $f_\bullet : C_\bullet \to D_\bullet$, is a collection of arrows $f_n : C_n \to D_n$ such the following diagram commutes.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & D_{n+1} & \xrightarrow[\partial'_{n+1}]{} & D_n & \xrightarrow[\partial'_n]{} & D_{n-1} & \longrightarrow & \cdots
\end{array}$$

Remark (V) 3.1.3. Note that for the above diagram to commute, it is sufficient and necessary for each square to commute and so we really require that $\partial'_n f_n = f_{n-1} \partial_n$. ◁

With that established, and the notion of composition of chain morphisms defined in the obvious manner, we write $\mathrm{Ch}(\mathcal{A})$ for the category of chain complexes over the abelian category $\mathcal{A}$.

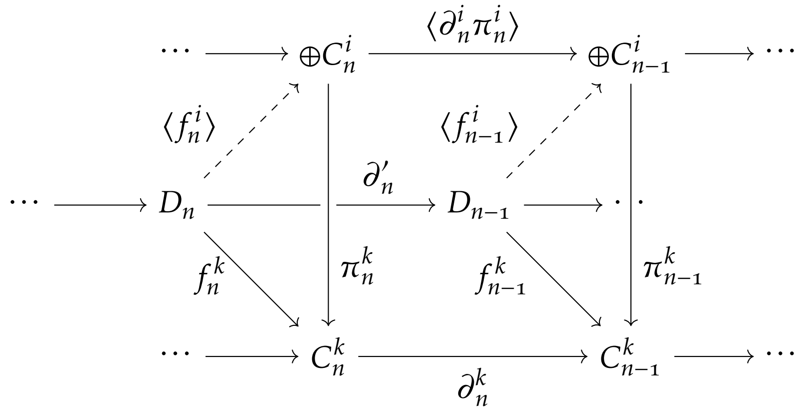
With the introduction of $\mathrm{Ch}(\mathcal{A})$, an entire line of inquiry becomes available and we may ask about the nature of $\mathrm{Ch}(\mathcal{A})$, and in particular, the extent to which the structure of $\mathcal{A}$ effects it.

Prop. (V) 3.1.4. *If $\mathcal{A}$ is abelian, then $\mathrm{Ch}(\mathcal{A})$ has all finite biproducts, kernels and cokernels, and they are computed 'degree-wise': $\oplus(C^i_\bullet, \partial^i) \cong (\oplus C^i_\bullet, \langle \partial^i_\bullet \pi^i_\bullet \rangle)$, etc.*

Proof. Recall that given a finite set I and a collection of chain complexes $(C^i_\bullet, \partial^i_\bullet)_{i \in I}$, $C^i_n \in \mathrm{Obj}\,\mathcal{A}$ for every $i \in I$, $n \in \mathbb{Z}$ so that $\oplus_I C^i_n$ exists as an object in $\mathcal{A}$. With this in hand, we show that $(\oplus C^i_\bullet, \langle \partial^i_\bullet \pi^i_\bullet \rangle)$ is a chain and supports the correct universal property to be $\prod(C^i_\bullet, \partial^i_\bullet)$. By dualisation it will follow that $\coprod(C^i_\bullet, \partial^i) \cong (\oplus C^i_\bullet, \langle \partial^i_\bullet \pi^i_\bullet \rangle)$ and thus we conclude the existence of biproducts in $\mathrm{Ch}(\mathcal{A})$. First, observe that

$$\pi^k_n \langle \partial^i_n \pi^i_n \rangle \langle \partial^i_{n+1} \pi^i_{n+1} \rangle = \partial^k_n \partial^k_{n+1} \pi^k_{n+1} = 0$$

so that by universal property $\langle \partial^i_n \pi^i_n \rangle \langle \partial^i_{n+1} \pi^i_{n+1} \rangle = 0$. Then, suppose there was a chain $(D_\bullet, \partial'_\bullet)$ with chain maps $f^i_\bullet : D_\bullet \to C^i_\bullet$, and consider the following diagram.

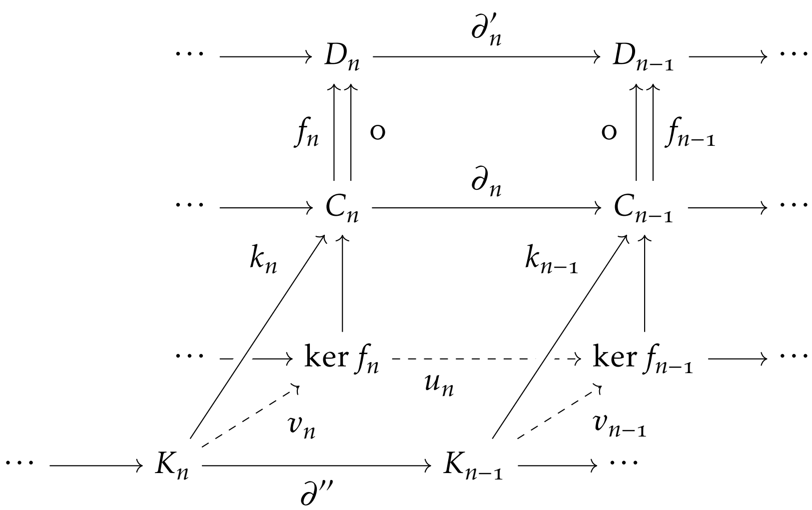


The bottom face commutes by assumption, and the left, right and front faces commute by definition of the maps $\langle f_n^i \rangle$, $\langle f_{n-1}^i \rangle$ and $\langle \partial_n^i \pi_n^i \rangle$ respectively. To see that the back face commutes, we must view the relevant composites as arrows $u_n : D_n \to \oplus C_{n-1}^i$ thereby uniquely characterising them by their projections. However,

$$\pi_{n-1}^k \langle \partial_n^i \pi_n^i \rangle \langle f_n^i \rangle = \partial_n^k f_n^k = f_{n-1}^k \partial'_n = \pi_{n-1}^k \langle f_{n-1}^i \rangle \partial'_n$$

The commutativity of the diagram thus follows and this, with dualisation, completes the proof of the existence of biproducts.

In an entirely similar vein, let $f_\bullet : C_\bullet \to D_\bullet$ be a chain morphism and suppose there was $k_\bullet : K_\bullet \to C_\bullet$ such that $f_n k_n = 0$ and consider the following diagram



Here we have noted that $f_{n-1} \partial_n \ker f_n = \partial'_n f_n \ker f_n = 0$ to find the unique arrow u_n with $(\ker f_{n-1}) u_n = \partial_n \ker f_n$, and v_n, v_{n-1} arise via assumption. With that established, we note that the top face commutes by assumption, and the left and right triangles and the central square commute by definition of the maps v_n, v_{n-1} and u_n respectively. To see that the bottom face commutes, that $u_n v_n = v_{n-1} \partial''_n$, we make use of what is effectively a universal property argument.

Observe that $\ker f_{n-1}$ is a monomorphism and post-composition of both composites yields the same morphism so that the entire diagram commutes. Explicitly,

$$(\ker f_{n-1}) u_n v_n = \partial_n (\ker f_n) v_n = \partial_n k_n = k_{n-1} \partial''_n = (\ker f_{n-1}) v_{n-1} \partial''_n$$

All that remains concerning $\ker f_\bullet$ is to show that $u_n u_{n+1} = 0$ making $(\ker f_\bullet, u_\bullet)$ a chain. This matter is quickly laid to rest when we recall that $\ker f_{n-1}$ is a monomorphism, so that the equality

$$(\ker f_{n-1})u_n u_{n+1} = \partial_n(\ker f_n)u_{n+1} = \partial_n\partial_{n+1}(\ker f_{n+1}) = 0 = (\ker f_{n-1})0$$

gives the required statement. Dualisation completes the proof. ■

This result allows us to tie the proverbial knot and demonstrate that

Cor. (V) 3.1.5. $\textsc{Ch}(\mathcal{A})$ *is abelian if* $\mathcal{A}$ *is abelian.*

Proof. Under the evident, degree-wise additive structure $\textsc{Ch}(A)$ is certainly pre-abelian due to the previous result. In order for the category to be considered abelian, it remains to be shown, as per prop. (V) 2.0.20, that for a chain morphism $f_\bullet : C_\bullet \to D_\bullet$, $\operatorname{im} f_\bullet \cong \operatorname{coim} f_\bullet$.

To begin, consider the collections $(\operatorname{im} f_\bullet, \widehat{\partial}_\bullet)$ and $(\operatorname{coim} f_\bullet, \widetilde{\partial}_\bullet)$ where the maps $\widehat{\partial}_\bullet$ and $\widetilde{\partial}_\bullet$ arise out of the following commutative diagrams.

$$\begin{array}{ccccc} C_n & \rightrightarrows & D_n & \longrightarrow & \operatorname{coker} f_n \\ {\scriptstyle \partial_n}\downarrow & & \downarrow{\scriptstyle \partial'_n} & & \downarrow{\scriptstyle \partial''_n} \\ C_{n-1} & \rightrightarrows & D_{n-1} & \rightarrow & \operatorname{coker} f_{n-1} \end{array} \qquad \begin{array}{ccccc} \ker f_n & \longrightarrow & C_n & \rightrightarrows & D_n \\ {\scriptstyle \partial'''_n}\downarrow & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial'_n} \\ \ker f_{n-1} & \longrightarrow & C_{n-1} & \rightrightarrows & D_{n-1} \end{array}$$

$$\begin{array}{ccccc} \operatorname{im} f_n & \longrightarrow & D_n & \rightrightarrows & \operatorname{coker} f_n \\ {\scriptstyle \widehat{\partial}_n}\downarrow & & \downarrow{\scriptstyle \partial'_n} & & \downarrow{\scriptstyle \partial''_n} \\ \operatorname{im} f_{n-1} & \longrightarrow & D_{n-1} & \rightrightarrows & \operatorname{coker} f_{n-1} \end{array} \qquad \begin{array}{ccccc} \ker f_n & \rightrightarrows & C_n & \longrightarrow & \operatorname{coim} f_n \\ {\scriptstyle \partial'''_n}\downarrow & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \widetilde{\partial}_n} \\ \ker f_{n-1} & \rightrightarrows & C_{n-1} & \rightarrow & \operatorname{coim} f_{n-1} \end{array}$$

To see that these are chains, consider that $\operatorname{coim} f_{n+1}$ is an epimorphism so we check $\widetilde{\partial}_n\widetilde{\partial}_{n+1}\operatorname{coim} f_{n+1} = \widetilde{\partial}_n(\operatorname{coim} f_n)\partial_{n+1} = (\operatorname{coim} f_{n-1})\partial_n\partial_{n+1} = 0 = 0(\operatorname{coim} f_{n+1})$ and dually for $\widehat{\partial}_n\widehat{\partial}_{n+1} = 0$. With that established, we must show that the following diagram commutes in order demonstrate that the degree-wise isomorphism extends to a chain isomorphism $(\operatorname{im} f_\bullet, \widehat{\partial}_\bullet) \cong (\operatorname{coim} f_\bullet, \widetilde{\partial}_\bullet)$, where $\overline{f}_i$ is the usual isomorphism in $\mathcal{A}$.

$$\begin{array}{ccc} \operatorname{im} f_n & \xrightarrow{\ \widehat{\partial}_n\ } & \operatorname{im} f_{n-1} \\ {\scriptstyle \overline{f}_n}\uparrow & & \uparrow{\scriptstyle \overline{f}_{n-1}} \\ \operatorname{coim} f_n & \xrightarrow[\ \widetilde{\partial}_n\ ]{} & \operatorname{coim} f_{n-1} \end{array}$$

We shall show that the two composites are equal by post-composition with $\operatorname{im} f_{n-1}$, a monomorphism.

First observe that $(\operatorname{im} f_{n-1})\widehat{\partial}_n\overline{f}_n = \partial'_n(\operatorname{im} f_n)\overline{f}_n$ by the bottom-left diagram. For the second composite we pre-compose with $\operatorname{coim} f_n$ to find $(\operatorname{im} f_{n-1})\overline{f}_{n-1}\widetilde{\partial}_n(\operatorname{coim} f_n) = (\operatorname{im} f_{n-1})\overline{f}_{n-1}(\operatorname{coim} f_{n-1})\partial_n = f_{n-1}\partial_n = \partial'_n f_n = \partial'_n(\operatorname{im} f_n)\overline{f}_n(\operatorname{coim} f_n)$ so that we may infer $(\operatorname{im} f_{n-1})\overline{f}_{n-1}\widetilde{\partial}_n = \partial'_n(\operatorname{im} f_n)\overline{f}_n$, thereby concluding the proof. ■

Of course, there are now many questions which we may ask about Ch($\mathcal{A}$). For example, under what circumstances does iteration of Ch produce genuinely new categories? If $\mathcal{A}$ is monoidal, is Ch($\mathcal{A}$) monoidal too? Is it closed? Regrettably, such investigations would divert our attention too extensively to admit discussion here.

Despite our demanding that $\partial\partial = 0$, in general a chain complex need not also be an exact sequence considered as a diagram in the underlying category. In fact, by and large, homology is the study of the deviation from exactness of such complexes. In particular,

Def. (V) 3.1.6. Given a chain complex $C_\bullet$ we define the n-th homology object to be $H_n(C) = \ker \partial_n / \operatorname{im} \partial_{n+1}$, understood in the generalised sense of prop. (V) 2.0.30. If $H_n \cong 0$ then we say that the complex is exact in degree n.

It is a simple matter to check that our terminology of exactness is warranted.

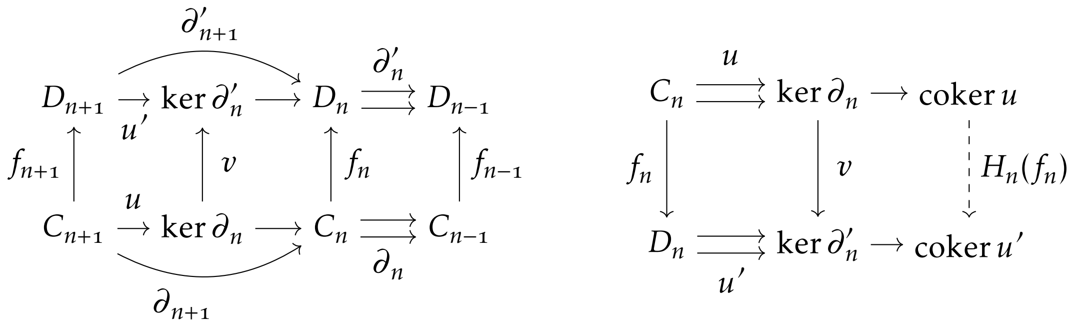
Prop. (V) 3.1.7. $H_n \cong 0 \implies \operatorname{im} \partial_{n+1} \cong \ker \partial_n$.

Proof. By prop. (V) 2.0.30, writing u for the unique arrow $\operatorname{im} \partial_{n+1} \to \ker \partial_n$, if $\operatorname{coker} u \cong 0$ then u must be epic (cor. (V) 2.0.21). However, by construction u is already monic and so u is an isomorphism (lem. (V) 2.0.19). ∎

Moreover, should we carefully view H_n as an assignment of objects from Ch($\mathcal{A}$) to $\mathcal{A}$, we may wonder whether H_n extends to a functor. Indeed,

Prop. (V) 3.1.8. *For each $n \in \mathbb{Z}$, $H_n : \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$ is a functor.*

Proof. Let $f_\bullet : (C_\bullet, \partial_\bullet) \to (D_\bullet, \partial'_\bullet)$, be a morphism of chain complexes. Recall that $H_n(C) = \ker \partial_n / \operatorname{im} \partial_{n+1} \cong \operatorname{coker}(\operatorname{im} \partial_{n+1} \to \ker \partial_n)$ (prop. (V) 2.0.30). Further, if we write $\partial_{n+1} = (\operatorname{im} \partial_{n+1})\widehat{\partial_{n+1}}$ for $\widehat{\partial_{n+1}}$ epic then it is apparent that we have isomorphisms $H_n(C) \cong \operatorname{coker}(\operatorname{im} \partial_{n+1} \to \ker \partial_n) \cong \operatorname{coker}(C_{n+1} \to \ker \partial_n)$ and so we construct the arrow $H_n(f_n) : \operatorname{coker}(C_{n+1} \to \ker \partial_n) \to \operatorname{coker}(D_{n+1} \to \ker \partial'_n)$.



In the above-left diagram, we have observed that $\partial_n \partial_{n+1} = 0$ to retrieve the arrow u and similarly u'. Then we noted that $\partial' f_n \ker \partial_n = f_{n-1} \partial_n \ker \partial_n = 0$ to find the arrow v. By the universal property of $\ker \partial'_n$, it must be the case that $v = u' f_{n+1} u$ and so the diagram commutes. Then, turning to the above-right diagram, observe that $(\operatorname{coker} u') v u = (\operatorname{coker} u') u' f_{n+1}$ as the above-left diagram commutes, to find the unique arrow $H_n(f_n)$ as desired. Given that we have defined $H_n(f_n)$ by universal property, it is readily apparent that $H_n(g_n f_n) = H_n(g_n) H_n(f_n)$ and it is a simple matter to see that $H_n(\mathrm{id}_n) = \mathrm{id}_{\operatorname{coker} u}$. Finally, to conclude the proof recall that we simply have domain and codomain isomorphisms on $H_n(f)$ so as to interpret it as an arrow $H_n(C) \to H_n(D)$. ∎

As has been indicated, many more topics and questions concerning H_n are immediately apparent, but we shall not have time to explore them.

3.2. The simplex category

The goal of this section is to introduce a means to discuss the classical notion of singular homology (with which some familiarity on the part of the reader is assumed) in a sufficiently general context. In particular, we do not wish to constrain our considerations to the case of classical singular homology – chains of free abelian groups generated by continuous functions from the standard topological simplexes $\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1,\ t_i \geq 0\}$ into the space of concern with the morphisms of the chain given by the alternating sum of face maps. Indeed, as we shall see, we shall generalise the notion of standard simplex in such a way that we will be able to realise this construction in a categorical manner such that we may work in any abelian category.

We begin by introducing a particular monoidal category that is of central concern.

Def. (V) 3.2.1. The augmented simplex category, denoted Δ_a, is the category whose objects are finite ordinals and whose morphisms are order preserving functions. The simplex category is the full subcategory of non-zero ordinals and is denoted Δ.

Remark (V) 3.2.2. When we speak of ordinals in this section we will follow the standard definition due to Von Neumann which gives $0 = \phi$, $1 = \{0\}$, *etc.* It can be shown that each well-ordered set is order-isomorphic to precisely one such set and so, in this way, we have canonically chosen a representative of each equivalence class. Thus we will write $1 + 1 = 2$ with direct equality, and so on. ◁

Recall that given two ordinals we may take their ordinal sum to arrive at a third ordinal, and that this sum is associative. Noting that there is an order isomorphism $n \cong \{\star\} \times n$, the ordinal sum $n + m$ is simply the usual coproduct of sets endowed with the evident total ordering that has $n < a$ for all $a \in m$. Consequently, $n = n + \phi = \phi + n$ and associativity of this operation can inductively be shown to be strict, where the base case is $(n + m) + \phi = n + (m + \phi)$. Moreover, $+$ extends to a map on morphisms in Δ_a as we define $f + f' : n + n' \to m + m'$ through

$$(f + f')(a) = \begin{cases} f(a), & a \in n \\ m + f'(a), & \text{otherwise} \end{cases}$$

It may be shown that this extension is functorial, making $+ : \Delta_a \times \Delta_a \to \Delta_a$ a bifunctor, thus rendering $(\Delta_a, 0, +)$ a strict monoidal category.

Def. (V) 3.2.3. Let $\delta^n_k : n \to n + 1$ be the injective order preserving function from n to $n + 1$ that omits $k \in n + 1$ in its image, $\delta_k(n) = \{0, \ldots, k - 1, k + 1, \ldots, n\} \subset n + 1$. Complementary to this, we write $\sigma^n_k : n + 1 \to n$ for the surjective order preserving function which does not increase on $k \in n + 1$, $\sigma^n_k(k) = \sigma^n_k(k + 1)$.

Remark (V) 3.2.4. Due to their geometric interpretation, the maps δ and σ are commonly referred to as the *coface* and *codegeneracy* maps. ◁

Prop. (V) 3.2.5. *The following identities hold*

1. $j < k \implies \delta_j^{n+1}\delta_k^n = \delta_{k+1}^{n+1}\delta_j^n$
2. $j \le k \implies \sigma_j^{n-1}\sigma_k^n = \sigma_k^{n-1}\sigma_{j+1}^n$
3. $\sigma_j^n\delta_k^n = \begin{cases} \delta_k^{n-1}\sigma_{j-1}^{n-1}, & k < j \\ \delta_{k-1}^{n-1}\sigma_j^{n-1}, & k > j+1 \\ \mathrm{id}_n, & (k=j) \lor (k=j+1) \end{cases}$

Proof. Each statement may be shown via direct computation. ■

Enabled by the calculus outlined above, it is a theorem of [ML97] that every arrow in Δ_a admits canonical decomposition in terms of δ and σ. Our interest in this fact is limited to stating that functors whose domain is Δ_a are determined by the objects in their image and their action on δ and σ alone. The particulars of the result are as follows.

Thm. (V) 3.2.6 (Mac Lane)**.** *Every arrow $f : n \to m$ in Δ_a admits a unique decomposition in terms of δ and σ as $f = \delta_{a_1} \circ \cdots \circ \delta_{a_k} \circ \sigma_{b_1} \circ \cdots \circ \sigma_{b_j}$ where $n + k = m + j$ and*

$$0 \le a_k < \cdots < a_1 < m, \quad 0 \le b_1 < \cdots < b_j < n-1$$

Regrettably, that is the limit of our interest in Δ_a specifically, but the reader may rest assured that augmentations and related concepts have found employ in the general theory – we mention them only for completion as our true interest lies in Δ. With that established, we introduce some terminology.

Def. (V) 3.2.7. Given a category $\mathfrak{C}$, an augmented simplicial object in $\mathfrak{C}$ is a functor $S : \Delta_a^{\mathrm{op}} \to \mathfrak{C}$. Correspondingly, a simplicial object in $\mathfrak{C}$ is a functor $S : \Delta^{\mathrm{op}} \to \mathfrak{C}$. A morphism of simplicial objects is a natural transform between the functors.

Though obvious, we nevertheless make explicit the relations that d and s satisfy as a result of being the functorial images of δ^{op} and σ^{op}, for later reference.

Cor. (V) 3.2.8 (Dual to prop. (V) 3.2.5)**.** *For a simplicial object $S : \Delta^{\mathrm{op}} \to \mathfrak{C}$, where $d = S\delta$ and $s = S\sigma$, the following identities hold.*

1. $j < k \implies d_j^{n-1}d_k^n = d_{k-1}^{n-1}d_j^n$
2. $j \le k \implies s_j^{n+1}s_k^n = s_{k+1}^{n+1}s_j^n$
3. $d_k^n s_j^n = \begin{cases} s_{j-1}^{n-1}d_k^{n-1}, & k < j \\ s_j^{n-1}d_{k-1}^{n-1}, & k > j+1 \\ \mathrm{id}_{Sn}, & (k=j) \lor (k=j+1) \end{cases}$

Example (V) 3.2.9

> Simplicial sets are presheaves on Δ, and form the category sSet.

With this particular example, we may attempt to shed some light on the nature of simplicial objects via simplicial sets and some geometric allegories.

Recall that the Yoneda embedding gives an embedding $h_- : \mathfrak{C} \to [\mathfrak{C}^{op}, \textsc{Set}]$ and so we may examine Δ under this embedding in $[\Delta^{op}, \textsc{Set}] = \text{s}\textsc{Set}$. In particular, let us consider the image of $n \in \Delta$ under this embedding and write Δ^n for $h_n = \Delta(-, n)$, which we shall term the *standard n-simplex*.

A convenient understanding of the standard n-simplex is as a generalised version of an ordered geometric simplicial complex. That is, we shall view Δ^n as a collection of sets (the images of the objects in Δ, no less) of geometric simplexes whose vertices are labelled by positive naturals ordered monotonically. These sets are indexed by the object of Δ in question (annoyingly this is one more than the *geometric* dimension, for two points make a line, *etc.*), $\Delta^n \sim \{S_1, S_2, \cdots\}$. However, we must also allow for 'degenerate' geometric simplexes in which some adjacent vertices coincide.

Explicitly, if we were to write out Δ^2 in this manner using the usual notation for simplexes on vertices, we would have sets $S_1 = \{[0], [1]\}$, $S_2 = \{[0,0], [0,1], [1,1]\}$, $S_3 = \{[0,0,0], [0,0,1], [0,1,1], [1,1,1]\}$, and so on – see fig. 5.1 for a visualisation. This notation is, obviously, doubly meaningful in this context. For Δ^n, the element $[v_1, \ldots, v_k] \in S_m$ with $k \in m$, $v_i \in n$, $i \leq j \implies v_i \leq v_j$, also records the order-preserving function in $\Delta(m, n)$. That is, we see $[v_0, \ldots, v_{m-1}]$ as the function which takes values $f(j) = v_j$.

Furthermore, this context allows for a convenient understanding of the face [1] and degeneracy maps. The face map is the image of δ_k under Δ^n and using our notation is an arrow $d_k = \Delta^n(\delta_k) : S_m \to S_{m-1}$ which takes a simplex and yields the embedded, 'lower dimensional' face by omitting the kth vertex, fig. 5.1. Really, however, this is just the usual precomposition of functions in $\Delta(m, n)$ by δ_k^{m-1}.

Dually, the degeneracy maps (images of σ under Δ^n) have type $S_m \to S_{m+1}$ and can be understood as taking a simplex $[v_0, \ldots, v_m]$ to the degenerate, 'higher dimensional' simplex $[v_0, \ldots, v_k, v_k, \ldots, v_m]$.

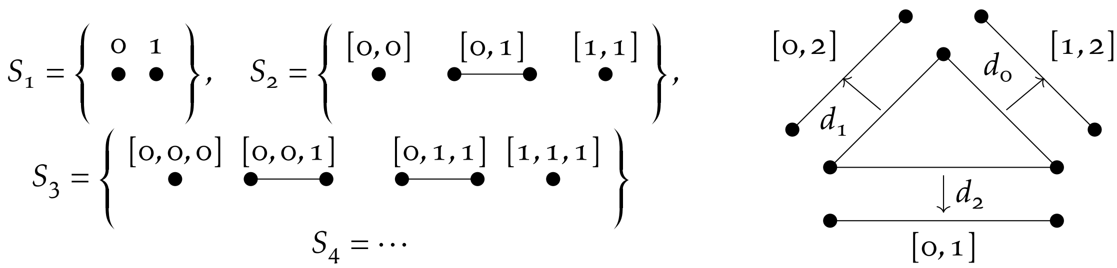


Figure 5.1: Left Δ^2 visualised, right face maps on $[0,1,2] \in S_3$ of Δ^3.

Though illuminating, as presented these notions only apply to the image of Δ embedded in s$\textsc{Set}$. More general simplicial sets are unlike the standard n-simplexes described above (in the extreme, consider the constant simplicial set). However, they still obey the same relations due to their functorial nature and so may be *thought* of in the same manner. In fact, in general, the above ideas serve as as an effective mental model of the underlying nature of simplicial objects in general categories – though this must be used with caution, certainly since 'elements' do not always have analogues.

[1] Recall that we are talking about presheaves and so Δ^{op}, hence face instead of *co*face.

With the idea of simplicial objects established, we are now in a position to recall the classical singular homology and attempt to phrase it in a far more general manner.

Simplicial Homology

Let us write $|\Delta^n|$ for the standard topological n-simplex given by the convex hull $\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1,\ t_i \geq 0\}$. Fixing a topological space X, it may be shown that the assignment $\mathcal{S} = \textsc{Top}(|\Delta^-|, X) : \Delta^{\mathrm{op}} \to \textsc{Set}$ which acts in the evident manner on objects and has $S(\delta_k^{\mathrm{op}})(f)$ as the restriction of f to the simplex without vertex k and likewise for sigma, is functorial and so forms a simplicial set. Thus, we have managed to phrase the selection of continuous maps from the standard geometric simplex into a space in terms of simplicial sets.

Remark (V) 3.2.10. That we have chosen to write $|\Delta^n|$ is no accident of notation. It is an indication that something far more interesting is happening behind the scenes. In general, there exists a way to construct a topological space from an arbitrary simplicial set, and moreover, this operation is functorial $|\cdot| : \mathrm{s}\textsc{Set} \to \textsc{Top}$ and is termed the *realisation* functor. Unfortunately, we will not have time to study realisations and the wonderful concepts to which they lead. ◁

Observe that, given a simplicial set $S : \Delta^{\mathrm{op}} \to \textsc{Set}$, we may post-compose the free abelian group functor $Z : \textsc{Set} \to \textsc{Ab}$ to arrive at a simplicial group. Moreover, had these sets been arrived at via $\textsc{Top}(|\Delta^n|, X)$, we would have a complete categorical version of the classical singular homology construction. Thus, should we recast the classical alternating face map result in a suitably general manner, we will have succeeded in categorifying singular homology. Of course, there is no reason to restrict ourselves to free groups, nor indeed Ab. Thus, the general version of the chain part of the construction of singular homology would read as follows.

Prop. (V) 3.2.11. *Let A be a simplicial object in an abelian category $\mathcal{A}$ with face maps d_k, and define $\partial_n : A_n \to A_{n-1}$ to be the map*

$$\partial_n = \sum_{k=0}^{n-1} (-1)^k d_k$$

then $(A_\bullet, \partial_\bullet)$ is a chain over $\mathcal{A}$.

Proof. We must show that $\partial_n \partial_{n+1} = 0$, and this may be achieved by the usual direct expansion, noting well cor. (V) 3.2.8 (1) which enables the pairwise term cancellation. ■

Thus, the categorical description of simplicial homology admits a neat summary.

Def. (V) 3.2.12. The simplicial homology of a simplicial object $A : \Delta^{\mathrm{op}} \to \mathcal{A}$, where $\mathcal{A}$ is an abelian category, is the chain homology of the chain $(A_\bullet, \partial_\bullet)$ where $\partial_\bullet$ is as in prop. (V) 3.2.11.

4. Regular categories

Although we managed to retrieve a great deal of algebraic notions of images and kernels in a more general manner, we did so in the setting of abelian categories.

In general it is undesirable to require the full force of abelianness (or even Ab enrichment), for example, to prove statements such as (but not limited to) unique factorisation of morphisms through their images. To this end, and for completion and to better flesh out the hierarchy presented, we achieve 'algebraic feeling' categories by introducing the following notion.

Def. (V) 4.0.1. Given an arrow $f : A \to B$ in $\mathfrak{C}$, the kernel pair of f is the pullback of f along itself, viz., $p_1, p_2 : P \rightrightarrows A$.

Remark (V) 4.0.2. In general categories, the kernel pair and the kernel are not isomorphic. In $\text{Set}_\bullet$, for example, the kernel of a morphism $f : (X, x) \to (Y, y)$ is the set $\{a \in X \mid f(a) = y\}$ whereas the kernel pair is the set $\{(a, a') \in X \times X \mid f(a) = f(a')\}$. The only "natural" map here is the diagonal inclusion of the former into the latter. In fact, this morphism exists in a general category, due to the universal property of the pullback. ◁

Prop. (V) 4.0.3. *If the kernel pair (P, p_1, p_2) of $f : A \to B$ exists, then p_1 and p_2 are epimorphisms.*

Proof. Note that $f\,\mathrm{id}_A = f\,\mathrm{id}_A$ and so by the universal property of the pullback there is a unique morphism $u : A \to P$ such that $p_i u = \mathrm{id}_A$. Thus u is a split monomorphism and p_i are split epimorphisms. ■

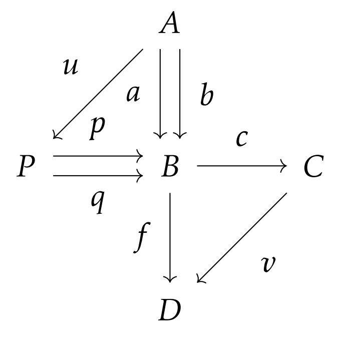
Prop. (V) 4.0.4. *The following conditions are equivalent for a morphism $f : A \to B$*

1. *f is a monomorphism*
2. *the kernel pair of f exists and is $(A, \mathrm{id}_A, \mathrm{id}_A)$*
3. *the kernel pair of f, (P, p_1, p_2) exists and has $p_1 = p_2$*

Proof. Assume that f is a monomorphism, and select a triple (C, c_1, c_2) with $fc_1 = fc_2$. We immediately have $c_1 = c_2$ and so may set $u = c_1 = c_2$ and uniqueness is evident. Then (2) obviously implies (3) and, assuming (3), given $a, b : D \rightrightarrows A$ with $fa = fb$ there is a unique arrow $u : D \to P$ with $b = up_i = a$. ■

Prop. (V) 4.0.5. *If a coequaliser has a kernel pair, then it is the coequaliser of its kernel pair. If a kernel pair has a coequaliser, then it is the kernel pair of its coequaliser.*

Proof. With the diagram below, consider the following.



Suppose $c = \text{coeq}(a,b)$ and (P,p,q) is its kernel pair. By the pullback property, we have $u : A \to B$ with $pu = a$ and $qu = b$. Suppose further that $f : B \to D$ has $fp = fq$ then we have $fa = fpu = fqu = fb$ and so a unique arrow $v : C \to D$, by the coequaliser property, with $vc = f$ making $c = \text{coeq}(p,q)$ by universal property.

Suppose now that (P,p,q) is the kernel pair of f and $c = \text{coeq}(p,q)$. By the coequaliser property, we have $v : C \to D$ with $vc = f$. Suppose further that the parallel arrows $a,b : A \rightrightarrows B$ have $ca = cb$, then we have that $fa = vca = vcb = fb$ and so we must have a unique arrow $u : A \to P$, by the pullback property, with $a = pu$ and $b = qu$, making (P,p,q) the kernel pair of c by universal property. ■

There is a final, technical result that we exhibit before addressing the matter at heart of this section.

Lem. (V) 4.0.6 (Pasting lemma for pullbacks)**.** *In the following commutative diagram, where the right-hand square is a pullback, the left-hand square is a pullback iff the outer square is a pullback.*

$$\begin{array}{ccccc} A & \xrightarrow{a_B} & B & \xrightarrow{b_C} & C \\ {\scriptstyle a_D}\downarrow & & \downarrow{\scriptstyle b_E} & & \downarrow{\scriptstyle c} \\ D & \xrightarrow[d]{} & E & \xrightarrow[e]{} & F \end{array}$$

Proof. Suppose that the left-hand square is a pullback and that there is a G with arrows $f : G \to C$ and $g : G \to D$ such that $cf = edg$. Then in particular we can view G has having arrows $f : G \to C$ and $dg : G \to E$ onto the right-hand square such that $cf = e(dg)$, so that there exists a unique $u : G \to B$ such that $b_C u = f$ and $b_E u = dg$. Then, we may view G as having arrows $g : G \to D$ and $u : G \to B$ onto the left-hand square such that $dg = b_E u$ and so by the pullback property there exists a unique arrow $v : G \to A$ such that $a_B v = u$ and $a_D v = g$. Consequently, $b_C a_B v = b_C u = f$ and $a_D v = g$ with $cf = edg$ and so the outer square is a pullback.

Now suppose that the outer square is a pullback and that there is a G with arrows $f : G \to C$ and $g : G \to D$ such that $cf = edg$. In this case, there exists a unique arrow $v : G \to A$ such that $b_C a_B v = f$ and $a_D v = g$. As such, G may be viewed as having arrows $f : G \to C$ and $dg : G \to E$ such that $cf = e(dg)$. Then, by the universal property of the pullback there exists a unique arrow $u : G \to B$ such that $b_C u = f$ and $b_E u = dg$. Observe that $a_B v = u$ by the uniqueness of this arrow, as $b_C a_B v = f$ and $b_E a_B v = d a_D v = dg$ by commutativity and universality of v. As such, G may be viewed as having arrows $g : G \to D$ and $u : G \to B$ such that $b_E u = dg$, where there exists a unique arrow $v : G \to A$ such that $a_D v = g$ and $a_B v = u$. ■

With that established, we now define a category that straddles the gap between being algebraic in a structural way and demonstrating desirable properties for certain objects.

Def. (V) 4.0.7. A category is regular if the following hold

1. every arrow has a kernel pair
2. every kernel pair has a coequaliser
3. the pullback of a regular epimorphism along any morphism exists and is again a regular epimorphism

Conveniently, and the author assures the reader here that this is no accident, we have a wealth of 'good' examples of regular categories. As we can see, abelian categories are, in particular, regular.

Cor. (V) 4.0.8. *In a regular category, the pullback of a composite of regular epimorphisms is again a composite of regular epimorphisms.*

Proof. Let $a : A \to B$ and $b : B \to C$ be regular epimorphisms and $f : D \to C$ be an arbitrary arrow. Consider the pullbacks

$$\begin{array}{ccc} P & \xrightarrow{p_B} & B \\ {\scriptstyle p_D}\downarrow & & \downarrow{\scriptstyle b} \\ D & \xrightarrow[f]{} & C \end{array} \qquad\qquad \begin{array}{ccc} Q & \xrightarrow{q_B} & A \\ {\scriptstyle q_P}\downarrow & & \downarrow{\scriptstyle a} \\ P & \xrightarrow[p_B]{} & B \end{array}$$

where p_D and q_P are regular epimorphisms because the category is regular. It is apparent that we can paste these two diagrams together, and so by lem. (V) 4.0.6 the outer square must also be a pullback and so $q_P p_D : Q \to D$ is the composite of two regular epimorphisms. ■

Prop. (V) 4.0.9. *If $\mathfrak{C}$ is regular, and the kernel pair of $f : X \to Y$ is $p_0, p_1 : P \rightrightarrows X$, with $c : X \to C = \operatorname{coeq}(p_0, p_1)$, then the unique arrow $v : C \to Y$ that arises from the coequaliser via f is a monomorphism. That is, the following diagram commutes.*

$$\begin{array}{ccccc} P & \overset{p_0}{\underset{p_1}{\rightrightarrows}} & X & \overset{c}{\twoheadrightarrow} & C \\ & & {\scriptstyle f}\downarrow & \swarrow{\scriptstyle v} & \\ & & Y & & \end{array}$$

Proof. Suppose there were parallel arrows $g, h : A \rightrightarrows C$ such that $vg = vh$. We begin by taking the following pullback,

$$\begin{array}{ccc} B & \xrightarrow{q_0 \times q_1} & X \times X \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle c \times c} \\ A & \xrightarrow[g \times h]{} & C \times C \end{array}$$

Observe that $fq_0 = vcq_0 = vga = vha = vcq_1 = fq_1$ and so we may derive a unique arrow $u : B \to P$ due to the universal property of the kernel pair, such that $p_0 u = q_0$ and $p_1 u = q_1$. With this in hand, we may state that $ga = cq_0 = cp_0 u = cp_1 u = cq_1 = ha$, where the middle equality arises from the coequaliser nature of c. If it were the case that a was an epimorphism, then we would have $g = h$ and the proof would be completed.

In order to demonstrate this, we decompose $c \times c$. Note that $c \times c$ is the composite of $c \times \mathrm{id}_X : X \times X \to C \times X$ and $\mathrm{id}_C \times c : C \times X \to C \times C$. Moreover, both of these morphisms are themselves pullbacks,

$$\begin{array}{ccc} X \times X & \xrightarrow{\pi_0} & X \\ {\scriptstyle c \times \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle c} \\ C \times X & \xrightarrow[\pi_0]{} & C \end{array} \qquad \begin{array}{ccc} C \times X & \xrightarrow{\pi_1} & X \\ {\scriptstyle \mathrm{id}_C \times c}\downarrow & & \downarrow{\scriptstyle c} \\ C \times C & \xrightarrow[\pi_1]{} & C \end{array}$$

Then, because $\mathfrak{C}$ is regular, both $c \times \mathrm{id}_X$ and $\mathrm{id}_C \times c$ are regular epimorphisms. As such, cor. (V) 4.0.8 informs us that $a : B \to A$ is thus composite of two regular epimorphisms and so, in particular, an epimorphism, and the result follows. ■

Prop. (V) 4.0.10. *If $\mathfrak{C}$ is regular then every arrow $f : X \to Y$ with image has that* $\mathrm{im}\, f \cong C = \mathrm{coeq}(p_0, p_1)$ *where $p_0, p_1 : P \rightrightarrows X$ is the kernel pair.*

Proof. Consider the following commuting diagram which we recover through the definition of $\mathrm{im}\, f$ and prop. (V) 4.0.9

$$\begin{array}{ccccc} P & \underset{p_1}{\overset{p_0}{\rightrightarrows}} & X & \xrightarrow{c} & C \\ & & {\scriptstyle h}\downarrow & \searrow^{f} & \downarrow{\scriptstyle v} \\ & & \mathrm{im}\, f & \xrightarrow[g]{} & Y \end{array}$$

Observe that $ghp_0 = fp_0 = fp_1 = ghp_1$ and so $hp_0 = hp_1$ as g is a monomorphism. Thus we recover a unique monomorphism $v : C \to \mathrm{im}\, f$ by the coequaliser property. Moreover, as $v : C \to Y$ is a subobject through which f factors, there must be a unique monomorphism $w : \mathrm{im}\, f \to C$ by the image property. The result follows with little effort. ■

Given this, in a regular category when we speak of image we will in fact be referring to the coequaliser of the kernel pair, as an appropriate generalisation of image.

Prop. (V) 4.0.11. *If $\mathfrak{C}$ is regular then every $f : X \to Y$ can be factored uniquely through its image as $f = ie$ with $i : \operatorname{im} f \to Y$ a monomorphism and $e : X \to \operatorname{im} f$ a regular epimorphism.*

Proof. Let $p_0, p_1 : P \rightrightarrows X$ be the kernel pair of f, and let $e : X \to \operatorname{im} f$ be its coequaliser. It is evident that e is a regular epimorphism and, from prop. (V) 4.0.9, that the unique arrow $i : \operatorname{im} f \to Y$ is a monomorphism.

For uniqueness, suppose $f = ie = i'e'$ with $i' : I \to Y$ a monomorphism and $e' : X \to I$ a regular epimorphism as the coequaliser of $k, l : C \rightrightarrows X$. Note that $i'e'p_1 = fp_1 = fp_0 = i'e'p_0$ and so $e'p_0 = e'p_1$ as i' is a monomorphism. Thus, by the coequaliser property of e we have a unique arrow $a : \operatorname{im} f \to I$ such that $e' = ae$. Similarly, as e' is a coequaliser and $iek = fk = fl = iel$ we have a unique arrow $b : I \to \operatorname{im} f$ such that $e = be'$. Consequently, $e = be' = bae$ and so $ba = \mathrm{id}_{\operatorname{im} f}$. Similarly, $e' = ae = abe'$ and so $ab = \mathrm{id}_I$. All that remains to be done is to note that $i'e' = i'ae = f = ie$ so that $i = i'a$. ■

With the statement of this proof, the reader should be fully convinced that regular categories afford us one of the key luxuries of abelian categories without requiring nearly as much of the category in question. In particular then, it should come as no surprise that

Thm. (V) 4.0.12. *All abelian categories are regular.*

Proof. Unfortunately, the proof of this matter would lead us too far astray, but the reader is encouraged to consult [Bor94] for details. ■

To do Actually, I could prove this, it depends on 1.7.6, 1.5.7, 1.5.4 but it would probably take too long. (20)

Moreover, we may wonder just how much of our discussion of exactness may be recovered in the regular case. After all, we do not have any obvious way to speak of sums or differences here. It may come as something of a surprise then that we are able to define and prove the following.

Def. (V) 4.0.13. In a regular category, a diagram of the form

$$P \underset{q}{\overset{p}{\rightrightarrows}} A \xrightarrow{f} B$$

is said to be exact when (P, p, q) is the kernel pair of f and $f = \operatorname{coeq}(p, q)$.

Observe that, in the above, f is a regular epimorphism and so by the regularity of the category, p and q are regular epimorphisms. Of course, given that we know that abelian categories are regular, we may at this point be wondering whether exactness as defined above coincides with the standard definition in that context. To answer this question, and end off this section, we cite a result given in [Bor94] which shows that it is indeed the case.

Thm. (V) 4.0.14 (Borceux). *In an abelian category,*

$$P \underset{q}{\overset{p}{\rightrightarrows}} A \xrightarrow{f} B$$

is exact (in the above sense) iff the following is a short exact sequence

$$0 \longrightarrow P \xrightarrow{\binom{u}{v}} A \oplus A \xrightarrow{(f \ \ -f)} B \longrightarrow 0$$

Thus we have seen that it is possible, on the shoulders of weaker assumptions, to recover the unique epi-mono factorisation of morphisms and even provide a means (though perhaps less generally useful) of discussing exact morphisms.

VI. Appendix

1. Elementary notions

1.1. Godement product

The observant reader may well note that, upon close inspection, there appear to be *two* ways in which we may compose natural transformations. If $\alpha : F \to G$ and $\beta : G \to H$ are natural, then we may well define $\beta\alpha$ with components $(\beta\alpha)_- = \beta_-\alpha_-$. This ordinary composition is known as *vertical* composition, for obvious reasons. However, if we have $\alpha : F \to G$ and $\beta : R \to S$ with R, S composable on F, G, then there ought to be a way to compose β with α to find a natural transformation $RF \to SG$. This to-be-defined composition is known as *horizontal* composition as we are composing along the direction of the functors, horizontally.

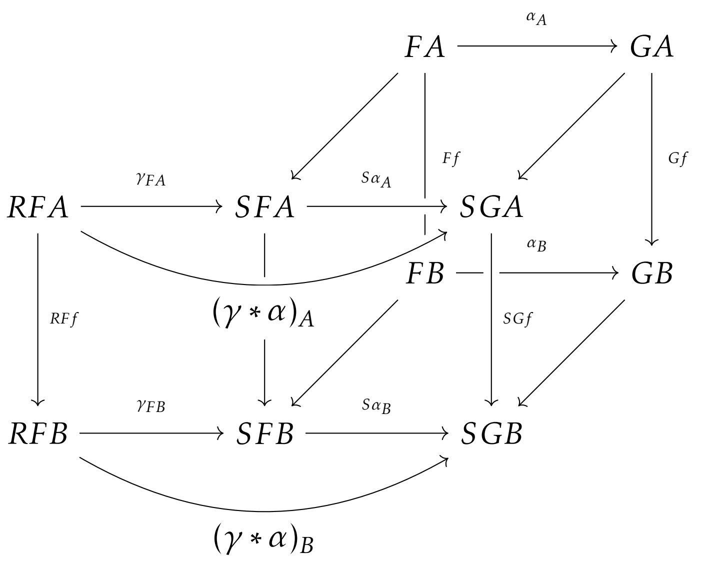
Def. (VI) 1.1.1. Given two pairs of parallel functors $F, G : \mathfrak{B} \rightrightarrows \mathfrak{C}$ and $R, S : \mathfrak{C} \rightrightarrows \mathfrak{D}$, and two natural transformations $\alpha : F \to G$ and $\gamma : R \to S$, we define the Godement product (horizontal composite) $\gamma * \alpha : RF \to SG$ to be the natural transform with components

$$(\gamma * \alpha)_B = \gamma_{GB} R\alpha_B = S\alpha_B \gamma_{FB}$$

Remark (VI) 1.1.2. It is a simple exercise to verify that the definition made above is valid. That the two expressions claimed to be equal are as such, follows simply from the naturality of γ (or α), that is for $B \in \operatorname{Obj} \mathfrak{B}$ we have

$$\begin{array}{ccc} RFB & \xrightarrow{\gamma_{FB}} & SFB \\ {\scriptstyle R\alpha_B}\big\downarrow & & \big\downarrow{\scriptstyle S\alpha_B} \\ RGB & \xrightarrow[\gamma_{GB}]{} & SGB \end{array}$$

To see that $\gamma*\alpha$ is indeed natural one need only consider the obvious naturality squares for $A, B \in \operatorname{Obj} \mathfrak{B}$ and $f : A \to B$ and paste them together.

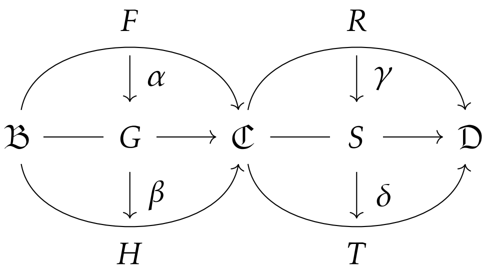


◁

Prop. (VI) 1.1.3. *The Godement product is associative.*

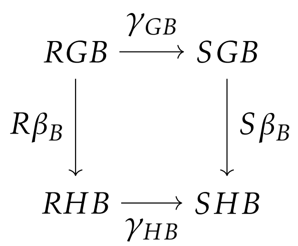
We will not belabour the proof here, as it is merely an exercise in drawing naturality squares and is neither illuminating nor beneficial. However, the doubtful or curious reader is encouraged to explore the details.

Prop. (VI) 1.1.4 (Interchange law)**.** *Given the following arrangement of functors and natural transforms*



it is the case that $(\delta \circ \gamma) * (\beta \circ \alpha) = (\delta * \beta) \circ (\gamma * \alpha)$ *as natural transformations* $RF \to TH$.

Proof. There is a large, 9×9 naturality diagram that may be drawn to fully appreciate the situation, but it is sufficient to consider the commuting diagram given below, together with the formulaic expansions of the quantities under consideration. Observe:



$$((\delta \circ \gamma) * (\beta \circ \alpha))_B = \delta_{HB}\gamma_{HB}R\beta_B R\alpha_B = \delta_{HB}S\beta_B\gamma_{GB}R\alpha_B = ((\delta * \beta) \circ (\gamma * \alpha))_B$$

■

In certain cases, the Godement product can be a clumsy formality whose meaning can be expressed in simpler terms. In particular, in a product containing id_F we simply write F itself and elide the intermediate $*$, for example, $\mathrm{id}_L * \tau * \mathrm{id}_R$ becomes $L\tau R$. This notation suggests of itself the behaviour of the composite natural transforms in that $(R\alpha)_B = R\alpha_B$ and and $(\gamma F)_B = \gamma_{FB}$.

In fact, this notation is suggestive of a 'factorisation law' governing our rules for juxtaposition of functors and natural transforms. Behold,

Cor. (VI) 1.1.5. *$(\alpha F)(\beta F) = (\alpha\beta)F$ and $(G\alpha)(G\beta) = G(\alpha\beta)$.*

Proof. Recall that $(\alpha F) \circ (\beta F) = (\alpha * \mathrm{id}_F) \circ (\beta * \mathrm{id}_F)$ so that a simple application of the interchange law gives $(\alpha \circ \beta) * (\mathrm{id}_F \circ \mathrm{id}_F) = (\alpha \circ \beta)F$. The second result follows similarly. ■

1.2. When are two things the same?

To do Equality, isomorphism, equivalence of categories (21)

1.3. Terribly boring things concerning limits and colimits

Def. (VI) 1.3.1. A category $\mathfrak{C}$ is finitely complete if it has all finite limits.

The careful reader may be wondering why the finiteness is emphasised so much in the definition, and indeed whether the case of 'full completeness', having literally all limits, is of any direct interest.

Surprise (VI) 1.3.2

> A category in which literally all limits exist is necessarily thin. The proof of this result is a rather simple cardinality argument. Suppose that literally all limits exist, and further that there are two, distinct, parallel morphisms $C \rightrightarrows D$ in the category. It is a simple matter to see that there must be $|2^P|$ morphisms $C \to \prod_P D$ for indexing set P. For the contradiction, set $P = \operatorname{Mor}\mathfrak{C}$ and note that $|P| < |2^P|$ to find that there cannot be two, distinct, parallel morphisms between any two given objects.

Prop. (VI) 1.3.3. *If a category has binary pullbacks and a terminal object, then it has binary products and equalisers.*

Proof. To find the product of $A, B \in \operatorname{Obj}\mathfrak{C}$, consider the pullback of $A \to 1 \leftarrow B$. It is apparent that if P has arrows to A and B that the extension to 1 equates the composites and so P has a morphism into the pullback. To find the equaliser of parallel arrows $f, g : A \rightrightarrows B$, consider first the pullback of $A \to B \leftarrow A$ to retrieve $p_1, p_2 : A \times_B A \rightrightarrows A$. By the universal property of $A \times A$, there is a unique morphism $\langle p_1, p_2 \rangle : A \times_B A \to A \times A$. Taking the pullback of $\langle p_1, p_2 \rangle$ and $\langle \mathrm{id}, \mathrm{id} \rangle$ readily gives the equaliser of f and g. ∎

Prop. (VI) 1.3.4. *A category is finitely complete iff it has binary products, equalisers and a terminal object.*

Proof. Note that the existence of binary products implies the existence of all finite products and that the forward implication is trivial. Then, in order to find the limit of a functor $F : \mathfrak{B} \to \mathfrak{C}$ with $\mathfrak{B}$ nonempty, we consider the equaliser of the following parallel morphisms

$$\alpha, \beta : \prod_{B \in \operatorname{Obj}\mathfrak{B}} FB \rightrightarrows \prod_{b \in \operatorname{Mor}\mathfrak{B}} F \operatorname{cod} b$$

where $\alpha_b = Fb\pi_{F\operatorname{dom} b}$ and $\beta_b = \pi_{F\operatorname{cod} b}$ induce α and β. We observe that the equaliser, $e : L \to \prod_{B \in \operatorname{Obj}\mathfrak{B}} FB$, must have $\alpha_b e = \beta_b e$ and so $Fb \circ F\operatorname{dom} b \circ e = F\operatorname{cod} b \circ e$ for every $b \in \operatorname{Mor}\mathfrak{B}$. Consequently $\phi_B = FBe$ is a cone for L, and it is a simple matter to see that it is the terminal cone, and thus the limit of F. If $\mathfrak{B}$ is empty, then the limit is the terminal object. ∎

Cor. (VI) 1.3.5. *A category is finitely complete iff it has binary pullbacks and a terminal object.*

2. Additive completions

A specific case of adjoint functors as they arise "in nature" is the following.

Def. (VI) 2.0.1. The Grothendieck group GM of a commutative monoid M is the abelian group such that there exists a monoid homomorphism $i : A \to G$ which is universal with respect to the following property

$$\begin{array}{ccc} M & \xrightarrow{i} & GM \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle u} & \\ A & & \end{array}$$

where A is an abelian group, f is a monoid homomorphism, and u is a group homomorphism.

Remark (VI) 2.0.2. There are several, equivalent ways to see that the Grothendieck group of a commutative monoid always exists, perhaps the most straightfoward of which is to define $G = M \times M/\sim$ where $(a,b) \sim (c,d) \iff (\exists k \in M)\, a + d + k = b + c + k$, and the group structure on the quotient as obvious. From this particular construction it is clear that M is cancellative iff $i : M \to G$ is injective. ◁

Given the above remark, we are motivated to rewrite the definition in terms of the 'forgetful' functor U : Ab → CMon so as to restate the Grothendieck group in terms of more familiar language. Thus, wee see that the Grothendieck group map on objects G : CMon → Ab has the following universal property

$$\begin{array}{ccccc} M & \xrightarrow{i} & UGM & & GM \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle Uu} & & & \downarrow{\scriptstyle u} \\ UA & & & & A \end{array}$$

and so, as we know, G extends to a functor and as a simple consequence,

Prop. (VI) 2.0.3. $G \dashv U$: CMon → Ab

Proof. Prop. (I) 3.0.2 (3). ■

To do This is all probably wrong given that the monoids are not cancellative! (22)

Prop. (VI) 2.0.4. *Every semi-additive category canonically gives rise to a related additive category on the same objects – the "additive completion".*

The proof of this matter is not especially difficult or interesting, but does involve many small computations and details. The reader is made aware of as many of these as seems reasonable, but routine computations are largely omitted.

Proof. Let $\mathfrak{C}$ be semi-additive, and define the additive completion $\mathfrak{C}^+$ to be the category defined to have objects $\operatorname{Obj}\mathfrak{C}^+ = \operatorname{Obj}\mathfrak{C}$ and morphisms $\mathfrak{C}^+(A,B) = G\mathfrak{C}(A,B)$, where $G \dashv U : \mathrm{CMon} \to \mathrm{Ab}$ is the Grothendieck group functor, and where we consider the morphism sets to bear the canonical commutative monoid structure (prop. (III) 2.0.17).

We first show that $\mathfrak{C}^+$ is a category. To do so, we define composition in $\mathfrak{C}^+$ to be the composite

$$\circ^+ = G\mathfrak{C}(B,C) \oplus G\mathfrak{C}(A,B) \xrightarrow{\tau} G(\mathfrak{C}(B,C) \oplus \mathfrak{C}(A,B)) \xrightarrow{G(i\circ)} G\mathfrak{C}(A,C)$$

where τ is the evident natural isomorphism (due effectively to the cocontinuity of G, dual of prop. (I) 3.0.8). In terms of the construction, we claim that this translates into the composite of the following maps

$$([(a,b)],[(c,d)]) \overset{\tau}{\mapsto} [((a,c),(b,d))] \overset{G(i\circ)}{\longmapsto} [(ac+bd, ad+bc)]$$

We must check that our given expressions of τ and $G(i\circ)$ are well defined and satisfy the necessary universal properties before concluding the definition of $\circ^+$.

In the case of the first map it is a simple exercise to verify that $[((a,c),(b,d))] = [((a',c'),(b',d'))] \iff ([(a,b)],[(c,d)]) = ([a',b'],[c',d'])$ and a straightforward exercise to check the naturality.

For the second, we check first that it is well defined. To do this, we show only that $((a,b),(c,d)) \sim ((a',b),(c',d))$ implies $(ab+cd, cb+ad) \sim (a'b+c', c'b+a'd)$ as the other implication follows symmetrically, and combined via transitivity they give that the map is well defined. To that end, consider that the data contained in $((a,b),(c,d)) \sim ((a',b),(c',d))$ is essentially the equation $a+c' = c+a'$. With this we note that $(a+c')b + (c+a')d = (c+a')b + (a+c')d \implies ab+cd+b'b+a'd = cb+ad+a'b+c'd \implies (ab+cd, cb+ad) \sim (a'b+c', c'b+a'd)$. Ergo the map is well defined.

To show that the map given is actually $G(i\circ)$ we must show that the correct universal property holds, that is, that the following diagram commutes.

$$\begin{array}{ccc}
\mathfrak{C}(B,C) \oplus \mathfrak{C}(A,B) & \xrightarrow{\quad i \quad} & UG(\mathfrak{C}(B,C) \oplus \mathfrak{C}(A,B)) \\
\circ \downarrow & & \downarrow UG(i\circ) \\
\mathfrak{C}(A,C) & \xrightarrow[\quad i \quad]{} & UG\mathfrak{C}(A,C)
\end{array}$$

To do so, we simply chase (g,f) about the diagram. Counter-clockwise we find $(g,f) \mapsto gf \mapsto [(gf,0)]$ and counterclockwise we have $(g,f) \mapsto [((g,f),(0,0))] \mapsto [(gf,0)]$ and so we conclude that $\circ^+$ is well defined.

Now that composition is defined, we must check that it is associative and unital in the proper manners. Through a straightforward computation on elements, associativity may be verified to hold as a result of the commutativity of addition and the distribution of composition (in $\mathfrak{C}$) over it. Finally, a quick computation shows that $[(\mathrm{id}_A, 0)]$ is respected as the identity under $\circ^+$ making $\mathfrak{C}^+$ into a category.

To complete the proof we must demonstrate that $\mathfrak{C}^+$ is Ab-enriched and has all finite biproducts. In order to show the first requirement we must demonstrate that $\circ^+$ is bilinear on the biproducts of morphism groups in $\mathfrak{C}^+$ as this would allow us to recast composition over tensor products and so give Ab-enrichment.

Verifying this amounts to showing that $n[(a,b)] \circ^+ [(c,d)] = [(a,b)] \circ^+ n[(c,d)]$ for any $n \in \mathbb{Z}$ and arbitrary composable arrows $[(a,b)]$ and $[(c,d)]$. Doing so is a straightforward exercise, but once again depends crucially on the distributivity of the underlying composition over addition (in particular, that $a(nc) = nac$).

To show that $\mathfrak{C}^+$ has all finite biproducts, we note that the *objects* $\bigoplus A_i$ still exist within C^+ and the map $i : \mathfrak{C}(A,B) \to U\mathfrak{C}^+(A,B)$ is a monomorphism. Thus, any relations between projections and inclusions in $\mathfrak{C}$ carry over (via i) to $\mathfrak{C}^+$. ■

Remark (VI) 2.0.5. There careful reader will note that our inspired choice for the definition of composition in the additive completion arises from

$$\begin{aligned} i\circ &\in \mathrm{CMon}\,(\mathfrak{C}(B,C)\oplus\mathfrak{C}(A,B), UG\mathfrak{C}(A,C)) \\ &\cong \mathrm{Ab}\,(G(\mathfrak{C}(B,C)\oplus\mathfrak{C}(A,B)), G\mathfrak{C}(A,C)) \\ &\cong \mathrm{Ab}\,(G\mathfrak{C}(B,C)\oplus G\mathfrak{C}(A,B), G\mathfrak{C}(A,C)) \ni \circ^+ \end{aligned}$$

where the first isomorphism is due to the adjunction $G \dashv U$ and the second is due to $\mathrm{Ab}(\tau, G\mathfrak{C}(A,C))$. ◁

Prop. (VI) 2.0.6. *Every semi-additive category admits a canonical functor to its additive completion, which is faithful and bijective on objects.*

Proof. Let $\mathfrak{C}$ be semi-additive and define $+ : \mathfrak{C} \to \mathfrak{C}^+$ to be the map $+A \mapsto A$ on objects, and $+f \mapsto if$ on morphisms. Note that although strictly speaking the codomain of i is a commutative monoid, it also has a group structure and so can equally well be seen as a group. Finally, it is apparent that $+$ is a functor and further that $+$ has the claimed properties as the morphism monoids in $\mathfrak{C}$ are cancellative. **To do** They are not necessarily cancellative! (23)

■

To do Part of an adjunction $^+ \dashv U : \mathrm{SAdd} \to \mathrm{Add}$? (24)

3. Rel, a treasure to behold

To do (25)

Def. (VI) 3.0.1. The category of relations, Rel, is defined to be the category whose objects are sets and whose morphisms $r : A \to B$ are relations $r \subseteq A \times B$. The identity morphisms are given by the diagonals $\mathrm{id}_A = \{(a,a') \in A \times A \,|\, a = a'\}$ and composition is simply composition of relations.

It is easy to verify that composition is associative and the the diagonal (discrete) relations are the units of composition. A first curious feature of Rel is enabled by the classical notion of the opposite relation. Recall that given a relation $r \subseteq A \times B$, we may form the opposite relation $r^{-1} = \{(b,a) \in B \times A \,|\, (a,b) \in r\} \subseteq B \times A$ and thus $r \in \mathrm{Rel}(A,B) \iff r^{-1} \in \mathrm{Rel}(B,A) = \mathrm{Rel}^{\mathrm{op}}(A,B)$. This simple observation all but proves the following result, which will be of some use later.

Prop. (VI) 3.0.2. $\mathrm{Rel} \cong \mathrm{Rel}^{\mathrm{op}}$

Proof. Define $F : \mathrm{Rel} \to \mathrm{Rel}^{\mathrm{op}}$ on objects to be the assignment$A \mapsto A$ and on morphisms $r : A \to B$ let Fr be the assignment $r \mapsto r^{-1}$, where r^{-1} is the opposite relation. As $(r^{-1})^{-1} = r$, clearly F admits an inverse assignment on morphisms (and trivially on objects) and so it remains to be shown that F is a functor.

Let $r : B \to C$ and $s : A \to B$ be arrows in Rel and expand definitions to find $F(r \circ s) = (r \circ s)^{-1} = s^{-1} \circ r^{-1} = r^{-1} \circ^{\mathrm{op}} s^{-1} = Fr \circ^{\mathrm{op}} Fs$ as desired. ■

Remark (VI) 3.0.3. Categories which are equivalent to their opposites are sometimes referred to as self-dual. ◃

This result ensures that whenever a limit exists in Rel, the associated colimit exists and *visa versa*. With that, a first step towards understanding the category is examining the nature of monomorphisms and epimorphisms in Rel. Of course, we need only obtain an explicit description of one these and the other may be obtained by passing to the opposite relation.

Prop. (VI) 3.0.4. *An arrow* $r : A \to B$ *in* Rel *is an epimorphism iff the following hold*

1. $\forall b \in B[\exists a \in A[(a,b) \in r]]$
2. $\forall a \in A[\forall b, b' \in B[\ (a,b) \in r \land (a,b') \in r \implies b = b']]$

Proof. Suppose $r : A \to B$ was an epimorphism, and that $(\exists b \in B)(\forall a \in A)\ (a,b) \notin r$. Consider then the arrows $p, q : B \rightrightarrows \{0,1\}$, given by $p = \{(b,0)\}$ and $q = \{(b,1)\}$. It is easy to see that $pr = \phi = qr$ but that $p \neq q$, a contradiction and thus the first condition holds. To see the second condition, suppose to the contrary that there exists some $a \in A$ for which there exist distinct $b, b' \in B$ with $(a,b), (a,b') \in r$. Let $p, q : B \rightrightarrows \{0\}$ be the distinct arrows $p = \{(b,0)\}$ and $q = \{(b',0)\}$, but we have that $pr = \{(a,0)\} = qr$, a contradiction.

For sufficiency of the conditions, suppose that there were arrows $p, q : B \rightrightarrows C$ such that $pr = qr$ and take $(b,c) \in p$. By assumption there exists an $a \in A$ such that $(a,b) \in r$ and so $(a,c) \in pr = qr$. Thus there exists a $b' \in B$ such that $(a,b') \in r \land (b',c) \in q$, but by assumption $b = b'$ so that $(b,c) \in q$ and $p \subseteq q$. Symmetrically we find $q \subseteq p$ and thus $p = q$. ■

Dualisation yields the description of monomorphisms in Rel.

Cor. (VI) 3.0.5. *An arrow $r : A \to B$ in* Rel *is a monomorphism iff the following hold*

1. $\forall a \in A[\exists b \in B[(a,b) \in r]]$

2. $\forall b \in B[\forall a, a' \in A[(a,b) \in r \wedge (a,b') \in r \implies a = a']]$

While trivial, this classification is nevertheless inelegant and is afforded a better form through the means of the following small observation.

Prop. (VI) 3.0.6. *The assignment $\mathcal{P}$:* Rel $\to$ Set *given by $\mathcal{P}A = [A, 2]$ on objects and $(\mathcal{P}s)A' = \{b \in B \mid \exists a \in A'[(a,b) \in s]\}$ for $s : A \to B$ and $A' \subseteq A$ extends to a faithful functor.*

Proof. It is immediately clear that $\mathcal{P}\mathrm{id}_A = \mathrm{id}_{\mathcal{P}A}$ and that $\mathcal{P}(rs) = \mathcal{P}r\mathcal{P}s$ holds follows from the definitions.

To see that $\mathcal{P}$ is faithful, suppose that $r, s : A \rightrightarrows B$ had $r \neq s$, and without loss of generality, specifically $r \setminus s \neq \phi$. For any $(a,b) \in r \setminus s$, it follows that $b \in \mathcal{P}r\{a\}$ but also that $b \notin \mathcal{P}s\{a\}$ and so $\mathcal{P}r \neq \mathcal{P}s$. ■

Note that, by construction, $\mathcal{P}r\phi = \phi$ for every relation r and so this functor is not full. We are now in a position to restate the result of prop. (VI) 3.0.4 more succinctly by means of $\mathcal{P}$.

Prop. (VI) 3.0.7. *An arrow $s : A \to B$ is an epimorphism iff the function $\mathcal{P}s$ is a surjection.*

Proof. This follows from definitions. ■

We shall return to this functor after establishing some results about limits in Rel.

An easy first result is in this direction is that Rel has a zero object, ϕ – of course, it has a zero object iff it has an initial object iff it has a terminal object as it has zero morphisms (the empty relation), and is also self-dual (either is sufficient in general). However, we can do better than this,

Prop. (VI) 3.0.8. Rel *has all small products.*

To do (26)

Before we prove this, however, a few small technical results will be of use.

Lem. (VI) 3.0.9. *Let $(A_i)_I$ be a collection of sets indexed by a set I, and $A = \bigcup(A_i \times \{i\})$ their disjoint union. If $\pi_i : A \to A_i$ denotes the relation $\pi_k = \{(a', a) \in A \times A_k \mid a' = \iota_k a\}$ where $\iota_k : A_k \to A$ are the usual set injections, then the following hold*

1. $\forall k, k' \in I[\forall (a, a') \in A_k \times A_{k'}[\iota_k a = \iota_{k'} a' \implies k = k' \wedge a = a']]$

2. $\forall a' \in A[\exists k \in I[\exists a \in A_k (a', a) \in \pi_k]]$

Proof. Both of these follow easily from definitions. ■

To do (27)

We are now in a position to prove prop. (VI) 3.0.8 via explicit construction.

Proof (prop. (VI) 3.0.8). Let $(A_i)_I$ be a collection of sets indexed by a set I, we will show that $(A = \bigcup(A_i \times \{i\}, \pi_i)$ with π_i as above is a product in Rel. First observe that $A \in \mathrm{Obj}\,\mathrm{Rel}$, so that it remains to be seen that (A, π_i) has the correct universal property for the product.

Let X be a set with arrows $f_i : X \to A_i$, and consider that we may construct $\langle f_i \rangle : X \to A$ as $\langle f_i \rangle = \{(x, a') \in X \times A \mid \exists k \in I\ [\exists a \in A_k[a' = \iota_k a \wedge (x, a) \in f_k]]\}$. We first verify that $\langle f_i \rangle$ has the correct projections

$$\begin{aligned}\pi_k \langle f_i \rangle &= \{(x, a) \in X \times A_k \mid \exists a' \in A[(x, a') \in \langle f_i \rangle \wedge (a', a) \in \pi_k]\} \\ &= \{(x, a) \in X \times A_k \mid \exists a' \in A[a' = \iota_k a \wedge \exists k' \in I\ [\exists a'' \in A_k[a' = \iota_k a'' \wedge (x, a'') \in f_k]]]\} \\ &= \{(x, a) \in X \times A_k \mid (x, a) \in f_k\} = f_k\end{aligned}$$

where we have made use of lem. (VI) 3.0.9 (1) in the last equality. With existence confirmed, we check uniqueness. Suppose that there was $g : X \to A$ with $\pi_k g = f_k$, and consider

$$\begin{aligned}(x, a') \in \langle f_i \rangle &\iff \exists k \in I[\exists a \in A_k[(a', a) \in \pi_k \wedge (x, a) \in f_k = \pi_k \langle f_i \rangle = \pi_k g]] \\ &\iff \exists k \in I[\exists a \in A_k[(a', a) \in \pi_k \wedge \exists a'' \in A[(x, a'') \in g \wedge (a'', a) \in \pi_k]]]\end{aligned}$$

lem. (VI) 3.0.9 (1)

$$\begin{aligned}&\iff \exists k \in I[\exists a \in A_k[(a', a) \in \pi_k \wedge (x, a') \in g]] \\ &\iff (x, a') \in g \wedge \exists k \in I[\exists a \in A_k (a', a) \in \pi_k] \iff (x, a') \in g\end{aligned}$$

where we have used lem. (VI) 3.0.9 (2) in the last transformation. Thus $\langle f \rangle = g$ and (A, π_i) is a product. ■

Using prop. (VI) 3.0.2 we can immediately conclude that Rel has all small biproducts – in fact, the biproducts here are strong in the sense that the isomorphism between product and coproduct (as constructed above) which gives the biproduct is an identity.

Cor. (VI) 3.0.10. Rel *is canonically enriched over* CMon.

Proof. Props. (VI) 3.0.2 and (III) 2.0.17. ■

Remark (VI) 3.0.11. It may entertain the reader to actually compute the commutative monoid structure. Recall that for parallel arrows $r, s : A \rightrightarrows B$, their sum was defined to be the composite $r + s = \nabla_B (r \oplus s) \Delta_A$, where $\Delta_A : A \to A \oplus A$ and $\nabla_B : B \oplus B \to B$ are the canonical codiagonal and diagonal arrows respectively and $r \oplus s = \langle r\pi, s\pi \rangle = [\iota r, \iota s]$. In our case, these specialise to the sets $\Delta_A = \{(a, a') \in A \times (A \oplus A) \mid a' = (a, 0) \vee a' = (a, 1)\}$, $\nabla_B = {\Delta_B}^{-1}$ and $s \oplus r = S \cup R$ where

$$\begin{aligned}R =& \{((a, 0), (b, 0)) \in (A \oplus A) \times (B \oplus B) \mid (a, b) \in r\} \\ S =& \{((a, 1), (b, 1)) \in (A \oplus A) \times (B \oplus B) \mid (a, b) \in s\}\end{aligned}$$

so that we ultimately find $s + r = s \cup r$. This obviously endows $\mathrm{Rel}(A, B)$ with a commutative monoid structure and as we well know leads to an enrichment of Rel, but the curious reader may wonder about the evident 'dual' structure on those same sets, viz., $s +' r = s \cap r$. While this is a valid structure for each morphism set individually, we note carefully that composition does not form a monoid homomorphism – a minimal counter-example is found by considering the relations $r = \{(a, 0)\}, s = \{(a, 1)\}$ and $t = \{(0, b), (1, b)\}$ so that $t(r \cap s) = \phi$ but $tr \cap ts = \{(a, b)\}$. ◂

With an understanding of the biproduct structure of Rel established, we return to the functor $\mathcal{P} : \textsc{Rel} \to \textsc{Set}$. Note that there is an obvious embedding of Set into Rel by given by taking each function to its graph, and so we have a pair of functors between Rel and Set.

While we have seen that $\mathcal{P}$ is not full and so this pair cannot be an isomorphism of categories, the possibility of the existence of an equivalence or adjunction. We check the latter first as it is a weaker condition, but we must first dispense with a question of asymmetry inherently present when dealing with adjunctions, viz., which functor could be the left adjoint?

Recall that, by prop. (I) 3.0.8, were G to be a right adjoint it would have to be continuous and so in particular $G1 = 1$ would have to be a terminal object in Rel; an immediate failing. On the other hand, this is no obstruction to the cocontinuity of G and a simple check reveals that the cardinality of the image of a biproduct matches the cardinality of the product of the images of the sets. We interpret this as an invitation and state

Prop. (VI) 3.0.12. $G \dashv \mathcal{P} : \textsc{Set} \to \textsc{Rel}$

Proof. We will give a binatural isomorphism. Let $\alpha_{A,B} : \textsc{Rel}(GA, B) \to \textsc{Set}(A, \mathcal{P}B)$ be the assignment $r \mapsto \alpha r$ where $\alpha r a = \mathcal{P}r\{a\} = \{b \in B | (a,b) \in r\}$ and conversely define $\beta_{A,B} : \textsc{Set}(A, \mathcal{P}B) \to \textsc{Rel}(GA, B)$ to be the assignment taking function f to the relation $\beta f = \{(a,b) \in A \times B | b \in fa\}$, that is, $(a,b) \in \beta f \iff b \in fa$.

With this established is a simple matter to verify that, for any function $f : A \to B$ and relation $r : A \to B$, $\alpha_{A,B}\beta_{A,B} f a = \{b \in B | (a,b) \in \beta f\} = \{b \in B | b \in fa\} = fa$ and $(a,b) \in \beta_{A,B}\alpha_{A,B} r \iff b \in \alpha_{A,B} r a = \{b \in B | (a,b) \in r\} \iff (a,b) \in r$ so that these assignments are inverses. Binaturality of these families of isomorphisms follows easily by expanding definitions, and in what follows we largely elide subscripts for clarity.

In the case of α we wish to show that, given a function $f : A' \to A$ and a relation $s : B \to B'$, we have the equalities $\textsc{Set}(A, \mathcal{P}s)\alpha_{A,B} = \alpha_{A,B'}\textsc{Rel}(GA, s)$ and $\textsc{Set}(f, \mathcal{P}B)\alpha_{A,B} = \alpha_{A',B}\textsc{Rel}(Gf, B)$.

Fixing a relation $r : A \to B$ and expanding the first to-be equality, we find that the left composite produces the function $\alpha(sr)$ whose action on elements is $\alpha(sr)a = \mathcal{P}(sr)\{a\}$. The right composite gives the function $\mathcal{P}s\alpha r$ whose action on elements is $\mathcal{P}s\alpha r a = \mathcal{P}s\mathcal{P}r\{a\} = \mathcal{P}(sr)\{a\}$ and so the equality holds. In a similar manner, the second to-be equality produces the functions $\alpha r f$ and $\alpha(rGf)$ from the left and right composites respectively. On arbitrary $a' \in A'$ we have the chain of equalities $\alpha(rGf)a' = \mathcal{P}(rGf)\{a'\} = \mathcal{P}r\mathcal{P}Gf\{a'\} = \mathcal{P}r\{fa'\} = \alpha r f a'$, thereby giving the naturality of α.

Then, the naturality of β requires that the identities $\textsc{Rel}(GA, r)\beta_{A,B} = \beta_{A,B'}\textsc{Set}(A, \mathcal{P}r)$ and $\textsc{Rel}(Gg, B)\beta_{A,B} = \beta_{A,B'}\textsc{Set}(g, \mathcal{P}B)$ hold, for arbitrary function $g : A' \to A$ and relation $r : B \to B'$. In the case of the first pair of composites, taking any $f : A \to \mathcal{P}B$ we find the relation $\beta(\mathcal{P}rf)$ defined by $(a,b') \in \beta(\mathcal{P}rf) \iff b' \in \mathcal{P}r(fa)$ where $Pr(fa) = \{\hat{b} \in B' | \exists b \in fa[(b,\hat{b}) \in r]\}$. Thus $(a,b') \in \beta(\mathcal{P}rf)$ iff $\exists b \in fa[(b,b') \in r]$ iff $\exists b \in B[(a,b) \in \beta f \wedge (b,b') \in r]$ but this is simply the statement $(a,b') \in r\beta f$, and so the first equality holds. For the second pair of composites, taking f as before, we find the relation $\beta f Gg$ defined by $(a',b) \in \beta f Gg \iff \exists a \in A[(a',a) \in Gg \wedge (a,b) \in \beta f]$. Using $(a',a) \in Gg \iff a = ga'$ and $(a,b) \in \beta f \iff b \in fa$ we see that $(a',b) \in \beta f Gg \iff b \in fga' \iff (a',b) \in \beta(fg)$, thereby concluding the proof. ∎

This result allows us to ask many questions of the relationship between the two categories.

To do (28)

To do (29) **To do** (30) **To do** (31)

To do (32) **To do** (33)

To do (34)

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To do...

- ☐ 1 (p. 8): Can this be phrased as an isomorphism of categories or objects?
- ☐ 2 (p. 9): Remark about explicit cue↔repr formulation, this should possibly come after adjunction.
- ☐ 3 (p. 16): Adjoint functor theorems
- ☐ 4 (p. 16): Reflective subcategories?
- ☐ 5 (p. 16): Free-forgetful adjunction
- ☐ 6 (p. 16): Anything that will prove useful later
- ☐ 7 (p. 16): Kleisli, Eilenburg-Moore?
- ☐ 8 (p. 21): Show this isomorphism is unique
- ☐ 9 (p. 35): This is probably false, or else needs to be checked for a monoidal isomorphism.
- ☐ 10 (p. 36): Free small category
- ☐ 11 (p. 39): Category object
- ☐ 12 (p. 39): Internal functor
- ☐ 13 (p. 39): Internal nat as right adjoint to product?
- ☐ 14 (p. 51): Strict 2-categories are Cat-enriched categories
- ☐ 15 (p. 51): Monads from 2-cat perspective
- ☐ 16 (p. 52): Horizontal categorification of monoidal
- ☐ 17 (p. 52): Use proofs from category object
- ☐ 18 (p. 52): Monad here is internal category
- ☐ 19 (p. 52): Work out what this is, and how a monad here is a category enriched over $\mathfrak{C}$.
- ☐ 20 (p. 86): Actually, I could prove this, it depends on 1.7.6, 1.5.7, 1.5.4 but it would probably take too long.

- ☐ 21 (p. 91): Equality, isomorphism, equivalence of categories
- ☐ 22 (p. 93): This is all probably wrong given that the monoids are not cancellative!
- ☐ 23 (p. 95): They are not necessarily cancellative!
- ☐ 24 (p. 95): Part of an adjunction $^{+} \dashv U : \text{SA}\textsc{dd} \to \textsc{Add}$?
- ☐ 25 (p. 96): Make mistake of defining objects to be relations, talk about philosophy.
- ☐ 26 (p. 97): Make mistake of products to be cartesian product, come back to this for monoidal.
- ☐ 27 (p. 97): Does a general coproduct in Set have the properties I want? n-lab only talks about using disjoint union...
- ☐ 28 (p. 100): Work out representability, free/forgetful, monad, algebra on monad, Kleisli category in this case. What else can we do with an adjunction?
- ☐ 29 (p. 100): Why is it not surprising that there are no (co)equalisers?
- ☐ 30 (p. 100): Axiom of choice here
- ☐ 31 (p. 100): Monoidal under $\times$
- ☐ 32 (p. 100): Enriched over posets
- ☐ 33 (p. 100): Posets = thin cats, so enriched over cat so strict 2-cat
- ☐ 34 (p. 100): What is a monad in Rel, hint, I think it's something nice.