Bi DOUBLING categories we'll see
MULTIPLE morphisms acting weakly
Two out of the four
have laws rather poor
But the rest is coherent VIRTUALLY!

a ramble for the JHU Category Theory seminar

by tslil clingman, 2020/09/16

(with apologies)

8 Bi DOUBLING categories we'll see
10 MULTIPLE morphisms acting weakly
x ✓ 5 Two out ? of the four
5 have laws ? rather poor
10 But the rest is coherent VIRTUALLY!

dibrach
iamb
spondee
anapest
amphibrach
dactyl
tribrach
erroneus

vs

da da DAH / da da DAH / da da THING
da da DAH / da da DAH / da da DING
da da DAH / da da OH
da da DAH / da da NO
da da DAH / da da DAH / da da SING
(pure anapest)

$$\frac{12+144+20+3\sqrt{4}}{7}+5\times 11=9^2+0$$

Table of contents:

$$
\begin{tikzcd}
 & \text{``you are here''} \arrow[dl] \arrow[dr] & \\
\text{``bi-way of analogy''} \arrow[d] & & \text{``the fork of Eratosthenes''} \arrow[d] \\
\text{``strange arrows of the second kind''} \arrow[dr] & & \text{``no complication without representation''} \arrow[dl] \\
 & \text{``towards a coherent vision of virtual reality''} &
\end{tikzcd}
$$

Outlook: this diagram is commutative

You are here

Recall monoidal categories: $(\mathcal{C}, \otimes, I, \rho, \lambda, \alpha)$ + laws

monoidal functors: $(F, \varphi_I, \varphi_{\bullet,\bullet})$ + laws (lax, strong, strict, ...)

let's keep in mind especially

$$(\mathrm{End}(\mathcal{C}), \circ, \mathrm{id}_{\mathcal{C}}, \dots)$$

embeds

$$\left(\mathrm{Psh}(\mathcal{C}^{op}\times\mathcal{C}), (F\otimes G)(a,b) := \int_{c\in\mathcal{C}} F(a,c)\times G(c,b), \ \mathcal{C}(-,-), \dots\right)$$

If this looks upside-down you might be in Australia

$$(\mathrm{Rel}(A), \text{relation composition}, \text{diagonal}, \dots)$$

$R \rightarrowtail A\times A$ objects

$\mathrm{Rel}(A)(R,S) = \begin{cases} \{*\}, & R\subseteq S \\ \emptyset, & \text{otherwise} \end{cases}$

embeds

$$(\mathrm{GenRel}(A), \text{pullback}, \text{diagonal}, \dots)$$

$X \xrightarrow{f} A\times A$ objects

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & A\times A \\ Y \arrow[ur, "g"'] & \end{tikzcd}$$ morphisms

$$({}_R\mathrm{Mod}_R, \otimes_R, R, \dots)$$

Questions: - The data seems reasonable enough, but why *these* laws?
- Are there others which give the "same" notion?
- If so, what *really* are monoidal categories?

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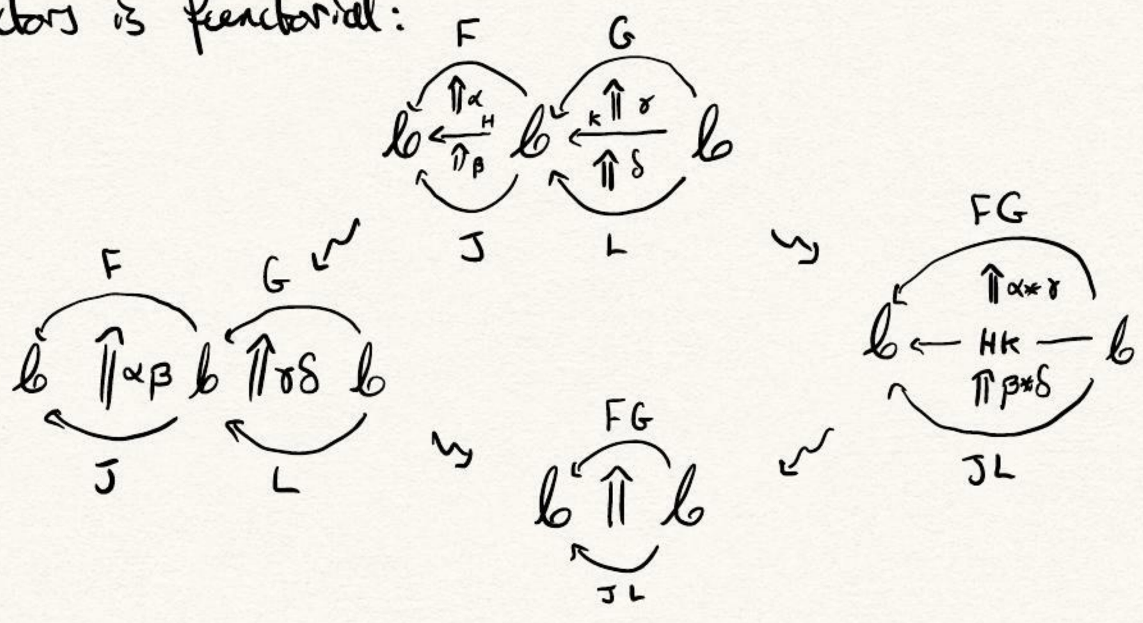
Bi-way of analogy

Question: If a monoid is a one-object category
then a monoidal category is a one-object ______?

We already have a hint. $(\mathrm{End}(\mathcal{C}), \circ, id_{\mathcal{C}}, \cdots)$ can be recovered by looking at the 2-category CAT of categories, functors, and natural transformations.

The monoidal structure carried by $(\mathrm{End}(\mathcal{C}), \circ, id_{\mathcal{C}}, \cdots)$ is a specialisation of the facts that $\mathrm{CAT}(\mathcal{C}, \mathcal{C})$ is a category and horizontal composition of functors is functorial:



In particular, $(\mathrm{End}(\mathcal{B}), \circ, id_{\mathcal{B}}, \dots)$ is a 1-object 2-category.

<u>BUT</u>: a 1-object 2-category is always a <u>strict</u> monoidal category.

→ What do we have to do to the notion of 2-category to extract general monoidal categories?

OR: "how to horizontally categorify"

=

<u>"Definition"</u>: A bi-category $\mathcal{K}$ comprises

objects: $A, B, \dots$ 1-cells: $A \xleftarrow{a} A'$ 2-cells:
$$\begin{tikzcd} A & B \arrow[l, bend right, "f"'] \arrow[l, bend left, "f'"] \arrow[l, phantom, "\Uparrow\alpha"] \end{tikzcd}$$

of which there are specified identity 1- and 2-cells

$$A \xleftarrow{A} A \qquad \begin{tikzcd} A & B \arrow[l, bend right, "f"'] \arrow[l, bend left, "f"] \arrow[l, phantom, "\Uparrow f"] \end{tikzcd}$$

along with operations of 1-cell composition $A \xleftarrow{f} B \xleftarrow{g} C \longmapsto A \xleftarrow{fg} C$

and 2-cell composition
$$\begin{tikzcd} A & B \arrow[l, bend right=60, "f"'] \arrow[l, "g" description] \arrow[l, bend left=60, "h"] \arrow[l, phantom, shift right=3, "\Uparrow\alpha"] \arrow[l, phantom, shift left=3, "\Uparrow\beta"] \end{tikzcd} \longmapsto \begin{tikzcd} A & B \arrow[l, bend right, "f"'] \arrow[l, bend left, "h"] \arrow[l, phantom, "\Uparrow\alpha\beta"] \end{tikzcd}$$

so that $\mathcal{K}(A,B)$ is a category,

and $\mathcal{K}(A,B) \times \mathcal{K}(B,C) \xrightarrow{\circ} \mathcal{K}(A,C)$ is a functor

<u>but</u> at the 1-level things hold only up to 2-isomorphisms:

$$\left\{ A \xleftarrow{f} B \xleftarrow{g} C \xleftarrow{h} D \right\} \quad \begin{tikzcd} A & D \arrow[l, bend right, "f(gh)"'] \arrow[l, bend left, "(fg)h"] \arrow[l, phantom, "\cong\Uparrow\alpha"] \end{tikzcd} \text{ and } \begin{tikzcd} A & B \arrow[l, bend right, "f"'] \arrow[l, bend left, "id_A \circ f"] \arrow[l, phantom, "\cong\Uparrow\lambda"] \end{tikzcd}, \begin{tikzcd} A & B \arrow[l, bend right, "f"'] \arrow[l, bend left, "f \circ id_B"] \arrow[l, phantom, "\cong\Uparrow\rho"] \end{tikzcd}$$

which are required to be "natural" and satisfy the same flavour of laws as monoidal categories.

=

Examples

Monoidal category	is a "hom" in the bi-category
$(\mathrm{End}(\mathcal{C}), \circ, \mathrm{id}_{\mathcal{C}}, \dots)$	CAT
$(\mathrm{Psh}(\mathcal{C}^{op} \times \mathcal{C}), F \otimes G, \mathcal{C}(-,-), \dots)$	Prof, where objects $\mathcal{C}$, 1-cells $\mathcal{C} \to \mathcal{D}$ are $F : \mathcal{C}^{op} \times \mathcal{D} \to \mathrm{SET}$
$(\mathrm{Rel}(A), \text{relation composition}, \text{diagonal}, \dots)$	Rel, where objects are sets, 1-cells $A \to B$ are $R \subseteq A \times B$, 2-cells $\subseteq$
$(\mathrm{GenRel}(A), \text{pullback}, \text{diagonal}, \dots)$	SPAN, where objets sets, 1-cells $A \to B$: $A \leftarrow X \rightarrow B$, 2-cells
$({}_R\mathrm{Mod}_R, \otimes_R, R, \dots)$	Ring, where objects R rings, 1-cells $R \xrightarrow{M} S$ bi-mods, 2-cells $R \Downarrow S$ bi-linear maps

"Bimod"
"Mod"

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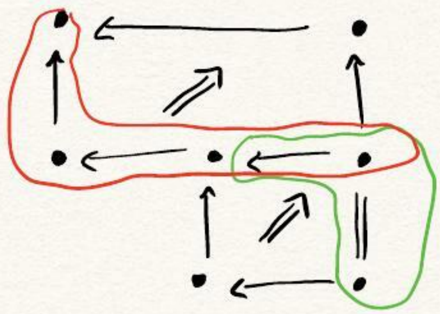
Unlike 2-categories, bi-categories pose a problem for pasting diagrams:



for instance, has no fixed meaning.

associativity

unitality

Are there multiple results?

"Theorem": Once we fix how we compose boundaries, there is a unique 2-cell result!

[See Thm 3.6.4 Johnson-Yau "2-dimensional categories" (20+ pages of work!)]

Monoidal functors also "categorify"

"Definition:" Given bi-categories $\mathcal{K}$ and $\mathcal{K}'$, a pseudo-functor $(F,\varphi): \mathcal{K} \longrightarrow \mathcal{K}$ comprises functors $F: \mathcal{K}(A,B) \longrightarrow \mathcal{K}'(FA,FB)$ as well as 2-isomorphisms

$$\begin{tikzcd} FA & FB \arrow[l, bend right, "F(fg)"'] \arrow[l, bend left, "(Ff)(Fg)"] \arrow[l, phantom, "\cong\Uparrow\varphi"] \end{tikzcd} \quad , \quad \begin{tikzcd} FA & FA \arrow[l, bend right, "F(id_A)"'] \arrow[l, bend left, "id_{FA}"] \arrow[l, phantom, "\cong\Uparrow\varphi_A"] \end{tikzcd}$$

which satisfy naturality + laws

Just like monoidal functors we can ask that

φ's are identities $\longrightarrow$ 2-functor

φ's are isos $\longrightarrow$ pseudo-functor

φ's are any 2-cells $\longrightarrow$ lax-functor

//

but lax-functors shouldn't exist. (fight me) 

Here are two examples to illustrate the problem:

1) a category enriched in a monoidal category $\mathcal{V}$ is the same thing as $\mathrm{Ind}_2 X \xrightarrow[\text{lax}]{} \Sigma \mathcal{V}$

$\mathrm{Ind}_2 X(x,y) = 1$

"suspension" one-object bicat

but ... all indiscrete 2-cats are equivalent

... is $\mathcal{V}$-CAT a point?

2) there are "pseudo-natural" transformations between pseudo-functors, but no notion of "___-natural" transformation is closed under whiskering by lax-functors

$\rightarrow$ No (good) n-category of bi-cats and lax-functors for $n > 1$.

$$\begin{tikzcd} \mathcal{K} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow\alpha"] & \mathcal{K}' \arrow[r, "F"] & \mathcal{K}'' \end{tikzcd}$$

We'll "fix" this later.

==

Strange arrows of the second kind

There's something missing in our world. Next time you find an algebraic geometer, ask them ⟨joke⟩.

We know that ring homorphisms exist, and we can study module maps over ring maps

$$\begin{tikzcd} R \arrow[d, "f"'] \arrow[dr, phantom, "\lfloor"] & S \arrow[l, "M"', "|" marking] \arrow[d, "g"] \\ R' & S' \arrow[l, "M'", "|" marking] \end{tikzcd}$$

We know that there a functions of sets, and we can study relation inclusions over functions

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, phantom, "\cap|"] & B \arrow[l, "R"', "|" marking] \arrow[d, "g"] \\ A' & B' \arrow[l, "R'", "|" marking] \end{tikzcd}$$

$\vdots$

Not by accident, every one of Prof, Rel, SPAN, Ring, has some other kind of morphism between its objects and 2-cells which can work over these.

$=$

""Definition:"" A pseudo-double category $\mathbb{D}$ comprises

objects: $A, B, \ldots$ arrows: $\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd}$ pro-arrows: $A \overset{P}{\nrightarrow} B$

squares/cells:

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, phantom, "\Theta"] & B \arrow[l, "P"', "|" marking] \arrow[d, "g"] \\ C & D \arrow[l, "Q", "|" marking] \end{tikzcd}$$

as well as

vertical identities and composition

$$A \\ \| \\ A \quad \mapsto \quad \begin{tikzcd} A \arrow[d, equal] \arrow[d, phantom, shift left=4ex, "\mathrm{vid}_p"] & A \arrow[l, "p"', "|" marking] \arrow[d, equal] \\ A & A \arrow[l, "p", "|" marking] \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, "a"'] \\ A' \arrow[d, "a'"'] \\ A'' \end{tikzcd} \mapsto \begin{tikzcd} A \arrow[d, "a'a"] \\ A' \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, "a"'] \arrow[dr, phantom, "\alpha"] & B \arrow[l, "p"', "|" marking] \arrow[d, "b"] \\ A' \arrow[d, "a'"'] \arrow[dr, phantom, "\beta"] & B' \arrow[l, "p'", "|" marking] \arrow[d, "b'"] \\ A'' & B'' \arrow[l, "p''", "|" marking] \end{tikzcd} \mapsto \begin{tikzcd} A \arrow[d, "a'a"'] \arrow[dr, phantom, "\begin{bmatrix}\alpha\\ \beta\end{bmatrix}"] & B \arrow[l, "p"', "|" marking] \arrow[d, "b'b"] \\ A'' & B'' \arrow[l, "p''", "|" marking] \end{tikzcd}$$

as well as horizontal identities and composition

...

so that pro-arrows and squares $\begin{tikzcd} \cdot \arrow[d, equal] & \cdot \arrow[l, "|" marking] \arrow[d, equal] \\ \cdot & \cdot \arrow[l, "|" marking] \end{tikzcd}$ form a bicategory

...

in a way that is compatible with vertical composition

...

and vertical identities

...

and this is suitably natural

...

<u>Note</u>: there isn't generally a vertical 2-category $\begin{tikzcd} \cdot \arrow[d] & \cdot \arrow[l, "|" marking] \arrow[d] \\ \cdot & \cdot \arrow[l, "|" marking] \end{tikzcd}$

=

<u>Examples</u>:

Prof:
$$\begin{tikzcd} \mathcal{C} \arrow[r, "P", "|" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha"] & \mathcal{D} \arrow[d, "G"] \\ \mathcal{C}' \arrow[r, "Q"', "|" marking] & \mathcal{D}' \end{tikzcd}$$
profunctors
functors
natural transformations

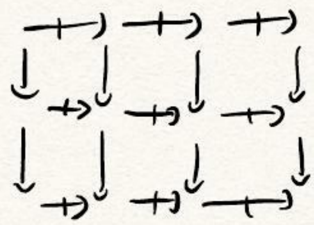
Ring:
$$\begin{tikzcd} R \arrow[r, "M", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "L"] & S \arrow[d, "g"] \\ R' \arrow[r, "N"', "|" marking] & S' \end{tikzcd}$$
bi-modules
ring maps
linear maps

Rel:
$$\begin{tikzcd} A \arrow[r, "R\subseteq A\times B", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\cap"] & B \arrow[d, "g"] \\ A' \arrow[r, "S\subseteq A'\times B'"', "|" marking] & B' \end{tikzcd}$$
relations
functions
inclusions

SPAN:
$$\begin{tikzcd} A \arrow[d, "f"'] & X \arrow[l, "a"'] \arrow[r, "b"] \arrow[d, "\alpha"] & B \arrow[d, "g"] \\ A' & Y \arrow[l, "a'"] \arrow[r, "b'"'] & B' \end{tikzcd}$$
spans
morphisms
span maps

If the horizontal bi-category is a 2-category then the whole thing is known as a double category.

Double categories have unambiguous composites



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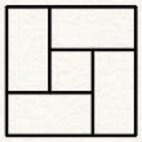
<u>but</u> not everything that is "composible" is composable



Dawson "A forbidden-suborder characterisation of binary-composable diagrams in double categories"

In general there should be some theorem for pseudo-double categories analogous to that for bicategories

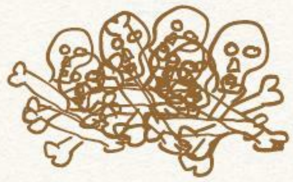
"fixing boundaries of a pasting* gives a unique square"

(where is it?)

* modulo these obstructions

=

morphisms of pseudo-double categories?



"The ability to reasonably fit the definition of an object in a talk" is probably a good litmus test.

Let's try again

You are here

Recall monoidal categories: $(\mathcal{C}, \otimes, I, \rho, \lambda, \alpha)$ + laws

monoidal functors: $(F, \varphi_I, \varphi_{\bullet,\bullet})$ + laws (lax, strong, strict, ...)

let's keep in mind especially $(\mathcal{C}, \amalg, 0, \ldots)$, $({}_R\mathrm{Mod}_R, \otimes_R, R, \ldots)$

Question: The data seems reasonable enough, but why these laws?

Are the others?

If so, and they are different, what really are monoidal categories?

//

The fork of Eratosthenes

We had previously thought of $(\mathrm{End}(\mathcal{C}), \circ, id_{\mathcal{C}}, \ldots)$ which gave us the impression that monoidal categories were shadows of higher dimensional composition structures (bi-categories, pseudo-double categories).

Let's rewind and map out this space some more.

Consider $(\mathcal{C}, \amalg, 0, \ldots)$, $\mathcal{C}(C \amalg D, E) \cong \mathcal{C}(C, E) \times \mathcal{C}(D, E)$

$({}_R\mathrm{Mod}_R, \otimes_R, R, \ldots)$ $\quad {}_R\mathrm{Mod}_R(M \otimes_R N, L) \cong$ R-multilinear maps $(M \times N \to L)$

so, by Yoneda, do we really need the "functors" $\amalg$ and $\otimes_R$ at all here?

what is primary?

//

No complication without representation

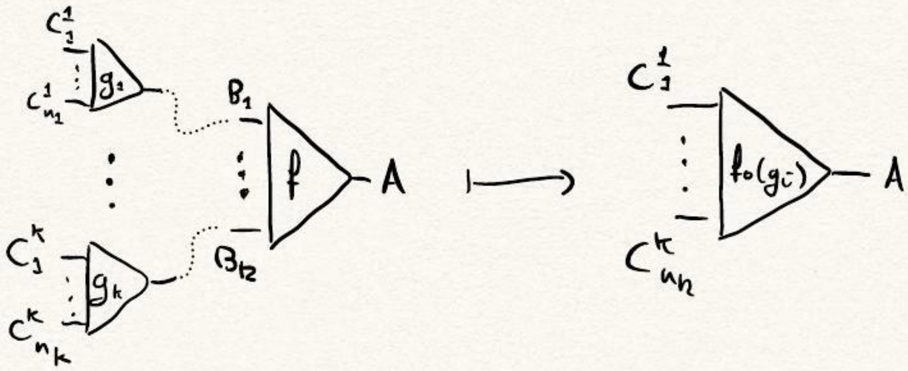
Let's aim for this "multi-linear" maps idea.

"Definition:" A multi-category $\mathbb{M}$ comprises the data of

objects: $A, B, \dots$ and multi-morphisms: $A_1, \dots, A_n \xrightarrow{f} B$

(including $n=0$, empty domain)

as well as identities $A = A$ and composition operations



which are "multi-associative" and unital.

Examples:

- every monoidal category $\mathcal{M}$ is a multi-category

$$\mathcal{M}(A_1, \dots, A_n ; B) := \mathcal{M}(\bigotimes A_i, B)$$

- every category $\mathcal{C}$ is a multi-category

$$\mathcal{C}(A_1, \dots A_n ; B) := \prod \mathcal{C}(A_i, B)$$

what just happened?

//

<u>IF</u> $\mathcal{C}$ has coproducts then the first and the second multicategories would "be the same". But it <u>didn't need them</u>, we could work with $\mathcal{C}$ the multicategory in precisely the same way!

→ we have virtualised the monoidal structure!

We use virtual here to indicate a question of representability,

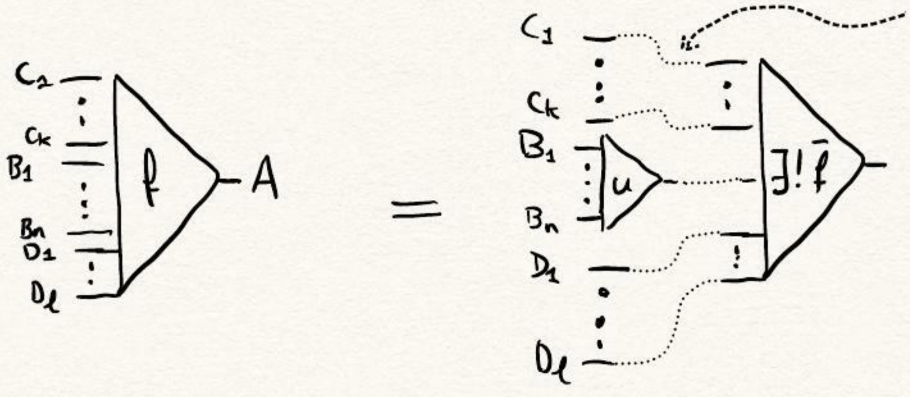
$$\mathcal{C}(A_1,\dots A_n;B) = \prod \mathcal{C}(A_i,B) \cong \mathcal{C}(\textstyle\coprod A_i ; B)$$

→ a representing object $\coprod A_i \in \mathcal{C}$
for multi-maps out of $A_1,\dots,A_n$.

How do we detect this <u>in</u> $\mathbb{M}$? Which are the multi-categories which are secretly monoidal?

=

"<u>Definition</u>": A representation for a list $B_1,\dots,B_n \in \mathbb{M}$ of objects is an object $\bigotimes B_i$ and a multi-morphism $u : B_1,\dots,B_n \longrightarrow \bigotimes B_i$ such that



(a universal property)

$$\forall k,\ \forall l,\ \forall f,\ \exists!\,\bar{f}$$

"Theorem": $\mathbb{M}$ "is a" monoidal category $\Leftrightarrow$ every object has a chosen representation

[Thm 3.3.4 Leinster "Higher Operads, Higher Categories"
// "pre-universal"]

multi-categories show that the laws of a monoidal category are presentations of a universal property!

//

But there's much more to multi-categories than this:

this is a category theory talk, i won't say the O-word

- one-object monoidal categories are commutative monoids and as multi-categories have just one multi-hom. One-object multi-categories are incredibly varied in general and underpin, for example, modern Alg. Top.

- multi-categories can encode all the multi-sorted strongly regular theories. They give rise to monads, and algebras for these are precisely models

- unlike monoidal categories where there is the tensor $A \otimes B$, we are now free to have many – closer to the real structure.
  (Span, ${}_R\mathrm{Mod}_R$, (co)cartesian, ...)

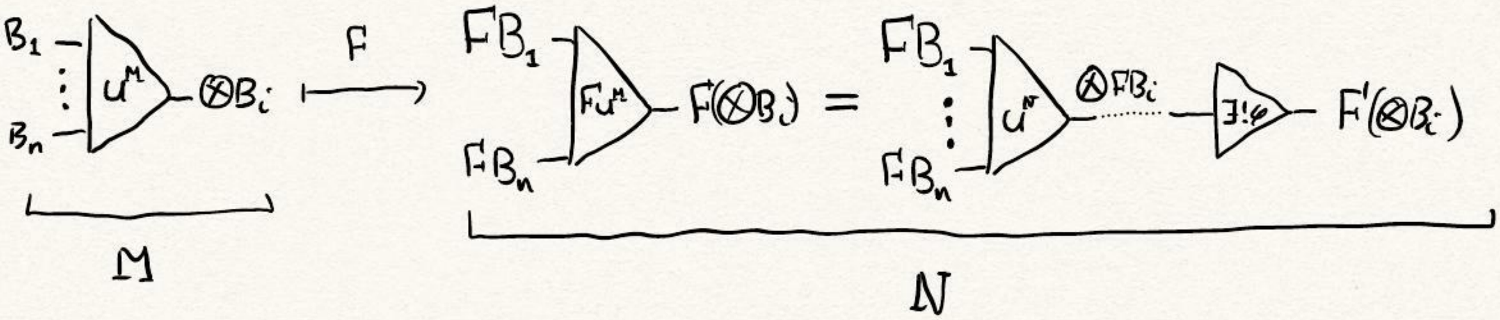
more to the point, multi-functors are just as clarifying

//

'Definition': A multi-functor of multi-categories, $F: \mathbb{M} \to \mathbb{N}$ is a collection of maps $\mathbb{M}(A_1, \dots, A_n; B) \to \mathbb{N}(FA_1, \dots, FA_n; FB)$ which preserve multi-compositions and identities.

Note: This is strict in every way!

What happens if both $\mathbb{M}$ and $\mathbb{N}$ have chosen representations for each object?



automatically derive maps $\varphi: \otimes FB_i \to F(\otimes B_i)$ which obey "all laws" by universal property! That is, F is lax-monoidal – but there's nothing "lax" about F.

//

Sends chosen reps. to	In monoidal world is
chosen reps.	strict - monoidal
reps.	strong - monoidal
nothing in particular	lax - monoidal

"monoidal strength = preservation of univ. prop."!

Towards a coherent vision of virtual reality

Question: If a monoid is a one-object category
then a ~~monoidal~~ multi category is a one-object ____ ?

we have the tools:

$A \xleftarrow{f} B$

$$\begin{tikzcd} * & * \arrow[l, bend right, "A"'] \arrow[l, bend left, "B"] \arrow[l, phantom, "\Downarrow f"] \end{tikzcd}$$

$$\begin{tikzcd} * \arrow[d, equal] \arrow[r, phantom, "f"] & * \arrow[l, "A"', "|" marking] \arrow[d, equal] \\ * & * \arrow[l, "B", "|" marking] \end{tikzcd}$$

$\mathcal{M}$ monoidal $\rightsquigarrow$ $\Sigma\mathcal{M}$ one-object bi-cat. $\rightsquigarrow$ $\Sigma\Sigma\mathcal{M}$ one-object vert. discrete pseudo-double cat.

$\mathbb{M}$ multi-cat. $\rightsquigarrow$ $\Sigma_1\mathbb{M}$ one-object ...? $\rightsquigarrow$ $\Sigma\Sigma_1\mathbb{M}$ one-object vert. discrete ...?

⟨remark⟩

$A \xleftarrow{f} B_1, \dots, B_n$

'**Definition**:' A virtual double category $\mathbb{X}$ comprises

objects: $A, B, \ldots$ vertical morphisms: $\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd}$ pro-arrows: $A \overset{p}{\leftarrow\!\!\!+} B$

cells: $\begin{tikzcd} A_1 \arrow[d, "f"'] & A_2 \arrow[l, "p_1"', "|" marking] & \cdots \arrow[l, "|" marking] & A_{n+1} \arrow[l, "p_n"', "|" marking] \arrow[d, "g"] \\ B_1 \arrow[rrr, phantom, "\alpha"] & & & B_2 \arrow[lll, "q", "|" marking] \end{tikzcd}$ including $n=0$, $\begin{tikzcd} & A_1 \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\alpha"] & \\ B_1 & & B_2 \arrow[ll, "q", "|" marking] \end{tikzcd}$

along with vertical identities $\begin{tikzcd} A \arrow[d, equal] \\ A \end{tikzcd}$, $\begin{tikzcd} A \arrow[d, equal] \arrow[r, phantom, "p"] & B \arrow[l, "p"', "|" marking] \arrow[d, equal] \\ A & B \arrow[l, "p", "|" marking] \end{tikzcd}$

and vertical compositions

$\begin{tikzcd} A \arrow[d, "g"'] \\ B \arrow[d, "f"'] \\ C \end{tikzcd} \longmapsto \begin{tikzcd} A \arrow[d, "fg"] \\ C \end{tikzcd}$,

$\begin{tikzcd} \bullet \arrow[d, "g_1^1"'] & \cdot \arrow[l, "p_1^1"', "|" marking] & \cdots & \bullet \arrow[l, "p_{k_1}^1"', "|" marking] \arrow[d, "g_2^1"] & \cdots & \bullet \arrow[d, "g_1^n"'] & \cdot \arrow[l, "p_1^n"', "|" marking] & \cdots & \bullet \arrow[l, "p_{k_n}^n"', "|" marking] \arrow[d, "g_2^n"] \\ \bullet \arrow[d, "f_1"'] \arrow[rrr, phantom, "\alpha^1", near start] & & & \bullet \arrow[lll, "q^1", "|" marking] & \cdots & \bullet \arrow[rrr, phantom, "\alpha^n", near start] & & & \bullet \arrow[lll, "q^n", "|" marking] \arrow[d, "f_2"] \\ \bullet \arrow[rrrrrrrr, phantom, "\beta", near start] & & & & & & & & \bullet \arrow[llllllll, "r", "|" marking] \end{tikzcd} \longmapsto \begin{tikzcd} \bullet \arrow[d, "f_1 g_1^1"'] & \cdot \arrow[l, "p_1^1"', "|" marking] & \cdots & \bullet \arrow[l, "p_{k_n}^n"', "|" marking] \arrow[d, "f_2 g_2^n"] \\ \bullet \arrow[rrr, phantom, "{\left[\begin{smallmatrix} \alpha^1 \cdots \alpha^n \\ \beta \end{smallmatrix}\right]}"] & & & \bullet \arrow[lll, "r", "|" marking] \end{tikzcd}$

which are multi-associative and unital.

__Note__: there is no horizontal composition, and cells always have a single target pro-arrow $\longrightarrow$ no pinwheel

$$\mathbb{P}\mathrm{rof}: \quad
\begin{tikzcd}
\mathcal{C}_1 \arrow[d, "F"'] & \cdots \arrow[l, "P_1"', "\shortmid" marking] & \mathcal{C}_{n+1} \arrow[l, "P_n"', "\shortmid" marking] \arrow[d, "G"] \\
\mathcal{D}_1 \arrow[rr, phantom, "\alpha", near start, yshift=3ex] & & \mathcal{D}_2 \arrow[ll, "Q", "\shortmid" marking]
\end{tikzcd}$$

pro-functors

functors

natural transf. $\prod P_i \to Q(F-,G-)$

$$\mathbb{R}\mathrm{ing}: \quad
\begin{tikzcd}
R_1 \arrow[d, "f"'] & \cdots \arrow[l, "M_1"', "\shortmid" marking] & R_{n+1} \arrow[l, "M_n"', "\shortmid" marking] \arrow[d, "g"] \\
S_1 \arrow[rr, phantom, "\mathcal{L}", near start, yshift=3ex] & & S_2 \arrow[ll, "N", "\shortmid" marking]
\end{tikzcd}$$

bi-modules

ring maps

multi-linear maps

or more generally, monoids and modules in somewhere

$$\mathbb{M}\mathrm{od}: \quad
\begin{tikzcd}
(X,\mu)_1 \arrow[d, "f"'] & \cdots \arrow[l, "M_1"', "\shortmid" marking] & (X,\mu)_{n+1} \arrow[l, "M_n"', "\shortmid" marking] \arrow[d, "g"] \\
(Y,\mu)_1 \arrow[rr, phantom, "\mathcal{L}", near start, yshift=3ex] & & (Y,\mu)_2 \arrow[ll, "N", "\shortmid" marking]
\end{tikzcd}$$

bi-modules

monoid morphisms

multi-natural maps

or $\mathcal{V}$-matrices (without assumptions on $\otimes$)

or $\mathbb{S}$PAN or $\mathbb{R}$el or functors and left adjoints

or as we shall see, all X-categories for

$X \in \{1, \text{bi}, \text{monoidal}, \text{multi}, \text{pseudo-double}, \ldots\}$

$\vdots$

see Leinster HOHC §5 or

Crutwell-Shulman "A unified framework for generalised multicategories"

Facts: - there is an analogue of representation in a multi-category for strings $A_1 \overset{p_1}{\nleftarrow} \cdots \overset{p_n}{\nleftarrow} A_{n+1}$ of pro-arrows. these are called composites and are "universal" cells

$$\begin{tikzcd}
A_1 & \cdots \arrow[l, "p_1"'] & A_{n+1} \arrow[l, "p_n"'] \\
A_1 \arrow[u, equal] & & A_{n+1} \arrow[u, equal] \arrow[ll, "\otimes p_i"] \arrow[ul, phantom, "\mu"]
\end{tikzcd}$$

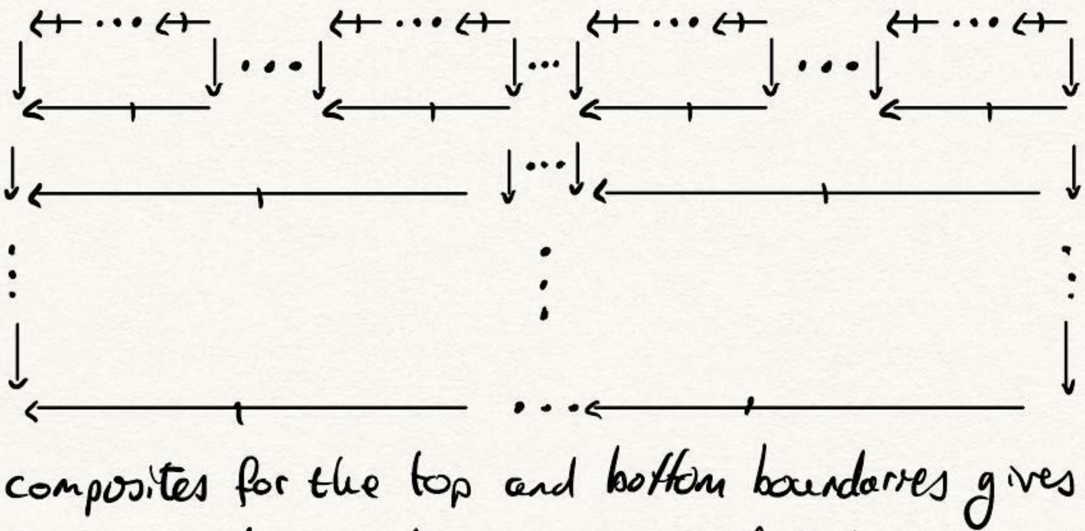
- pseudo-double category ⟺ virtual double category in which every string of pro-arrows has a chosen composite.

bi-category ⟺ vert. discrete + chosen composites

monoidal ⟺ one-object, vert. discrete + chosen comps

- the theorem about pastings in a vdbl. cat is "almost trivial"

given



choosing composites for the top and bottom boundaries gives directly, by universal properties, a unique cell between them!

- there are virtual double functors $\mathbb{X} \to \mathbb{Y}$ which are strict in every way, and if $\mathbb{X}$ and $\mathbb{Y}$ have chosen comps. then

Sends chosen comps. to	In bi-cat world is
chosen comps.	2 - functor
comps.	pseudo - functor
nothing in particular	lax - functor

⚠

lax functors <u>are</u> strict and <u>are</u> still functors! We weren't looking at the correct place!

this "fixes" everything

- If we repeat the $\mathrm{Ind}_2 X \xrightarrow{\text{lax}} \Sigma \mathcal{V}$ description for virtual double categories we find

"set X enriched in $\mathcal{V}$ monoidal" is

$$\mathrm{Ind}_2 X \xrightarrow[\text{functor}]{\text{vdbl.}} \Sigma_1 \Sigma \mathcal{V}$$

vert. discrete + single cell for every boundary

one-object vert. discrete

<u>but</u>

for $X \not\simeq Y$ there aren't any "vdbl. equivs"

$$\mathrm{Ind}_2 X \xrightarrow{\simeq} \mathrm{Ind}_2 Y$$

because those "only know about" the vertical direction and $x \mapsto y \mapsto x'$ is incomparable unless $x = x'$!

- similarly, there is a good virtual double category of virtual double functors and "whiskering" is not problematic there.

As we'll see in later talks vdbl is the 'correct' backdrop against which to do category theory - the theory of categories.

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Morals:

- although 2 is a difficult number, we have the technology

- (non-strict) composition is more readily understood as a universal property than as mysterious laws on operations

- functoriality is "really" about preservation of these universal properties

- if we see lax-type things, it's a good indication that something else is preserved strictly.

- there is a lot of low-hanging fruit in virtual double categories and virtual equipments (next time!), allow me to invite you to explore this powerfully unifying landscape!