The univalence axiom, a brief & incomplete tour

School and Workshop on Univalent Mathematics

tslil clingman, 2019

Univalence & function extensionality

Univalence, the structure-identity principle, and you

Open questions the formulation of univalence

Equivalent reformulations

Notation

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Univalence & function extensionality

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Types at issue

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We will prove that from a term ua we may deduce funext.

Proof idea

Let us fix $A : \mathsf{U_i}$, $B : A \to \mathsf{U_j}$, and $f : \prod(a : A), B$.

$$\sum_{(g : \prod(a : A), B)} f \sim g$$

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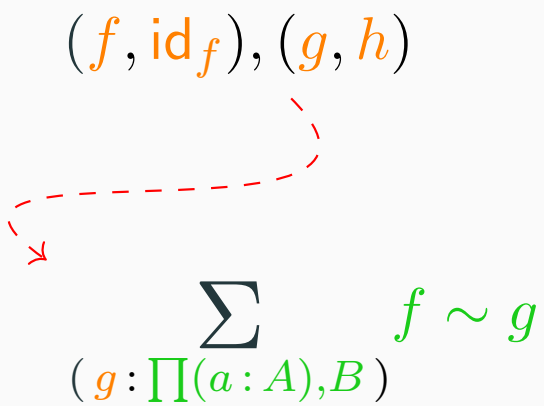

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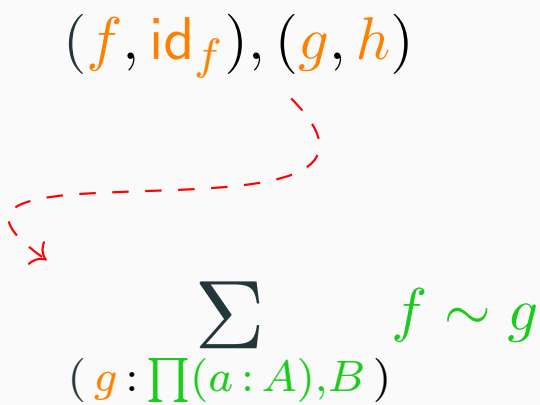

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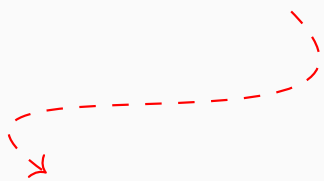

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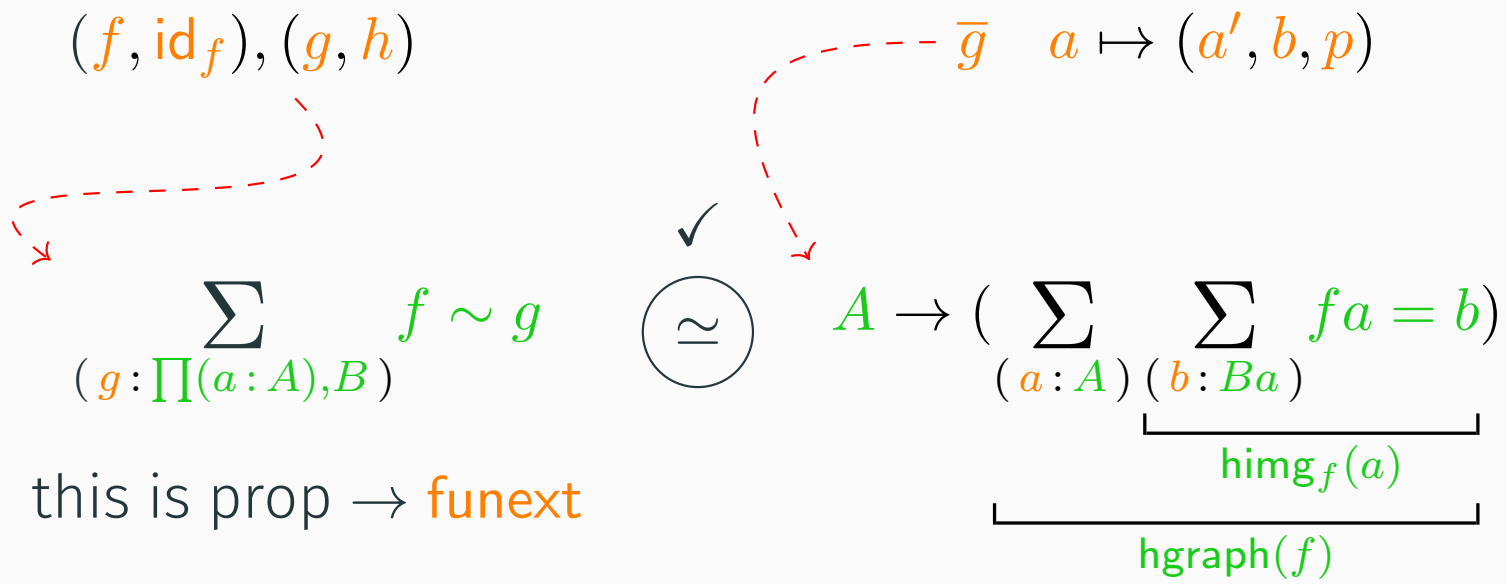

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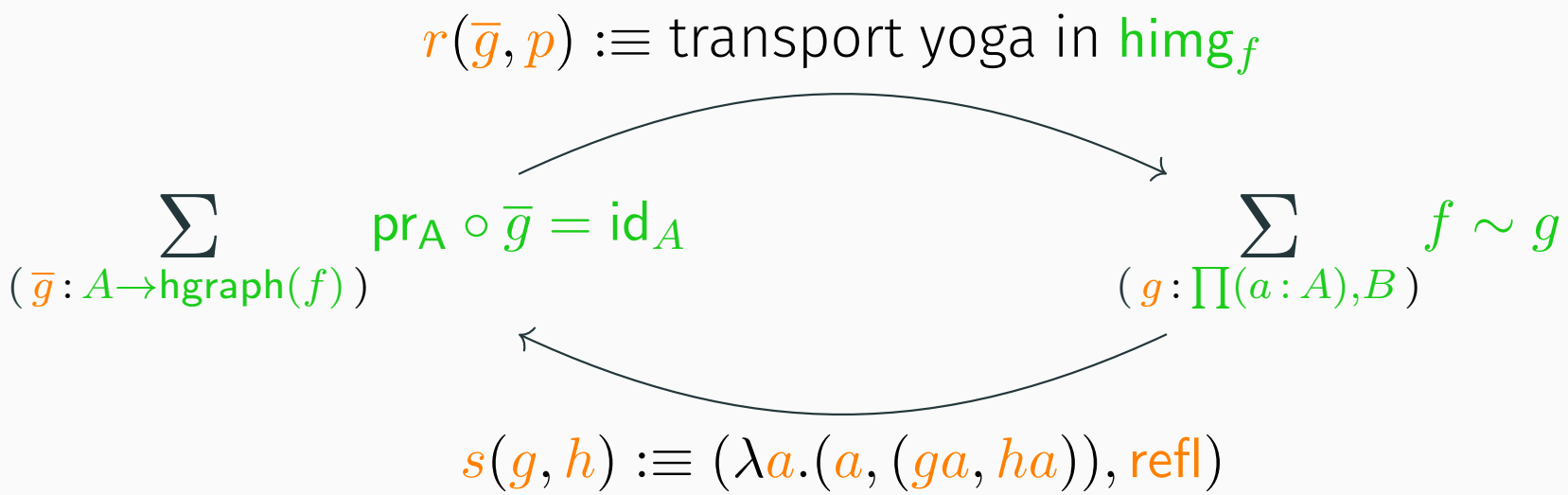

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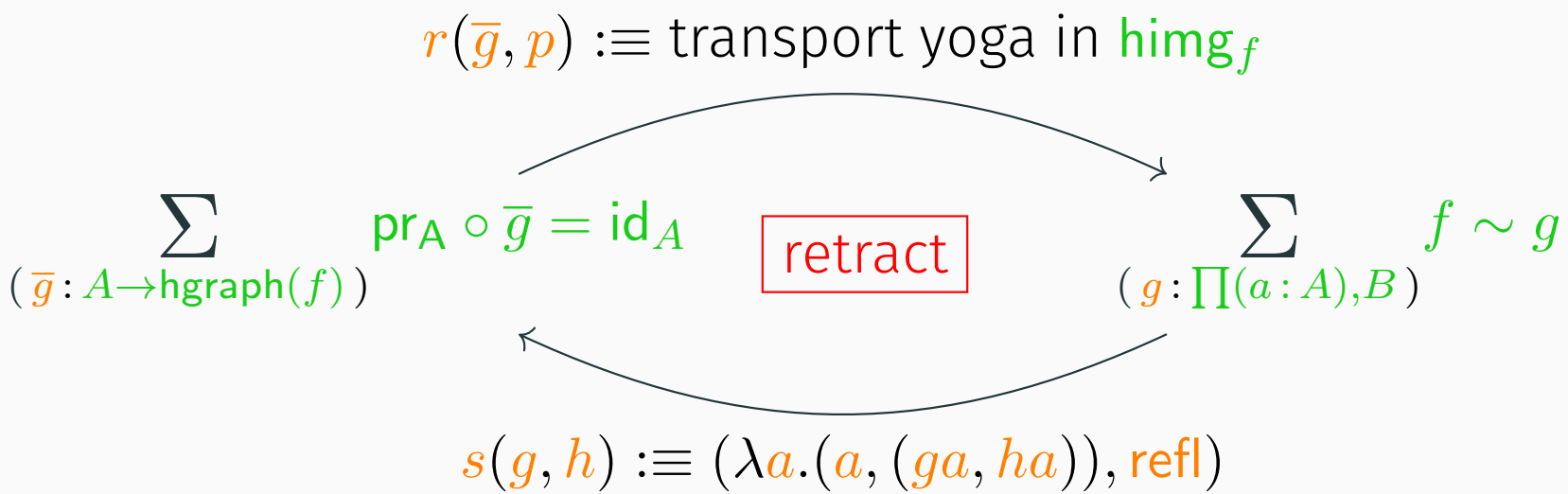

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$\sim$ 50 of which is a single, uninteresting, technical fact about equivalences

Is the aim of type theory is to make that which is formal, formally formal?

This proof is not intricate, most complicated part is some 2-path algebra.

I have formalised it...ahem in that other language.

$\sim$ 200 lines on the HoTT library,

$\sim$ 50 of which is a single, uninteresting, technical fact about equivalences $\rightsquigarrow$ good exercise?

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- Implies function extensionality

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- Consequently, for any proposition $P : \mathsf{U} \to \mathsf{U}$ and witness $A \simeq B$, one cannot show $P(A)$ but not $P(B)$.
- By the *structure identity principle*, ua means that many algebraic things are univalent too.

Univalence, the structure-identity principle, and you

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OR

Change foundations

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...where does the S.I.P. fail?

Open questions the formulation of univalence

Putting the *fun* in function extensionality

Definition (Naïve function extensionality)

$$\mathsf{nfunext} : \prod_{(A : \mathsf{U_i})} \prod_{(B : \mathsf{U_j})} \prod_{(f,g : \prod(a:A),B)} f \sim g \to f = g$$

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Can we do better?

The setup

Theorem

From nfunext we may derive funext.

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It is known that nua+nuaβ $\leftrightarrow$ ua

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Equivalent reformulations

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and this is logically equivalent to ua.

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1. unit : $A = \sum(a : A), 1$
2. flip : $\sum(a : A), \sum(b : B), C = \sum(b : B), \sum(a : A), C$
3. contract : isContr$(A) \to (A = 1)$
4. unitfi : $(\mathsf{coe}(\mathsf{unit}))a = (a, *)$
5. flipfi : $(\mathsf{coe}(\mathsf{flip}))(a, b, c) = (b, a, c)$

and this is logically equivalent to ua. So we can rephrase,

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Question

In the presence of funext, do (1.)-(3.) above imply (4.) and (5.), for possibly 'modified' unit and flip.