Metric Spaces Worksheet 3

Sequences II

We're about to state an important fact about convergent sequences in metric spaces which justifies our use of the notation $\lim a_n = a$ earlier, but before we do that we need a result about M2 – the separation axiom.

Lemma 1 (only equal points are arbitrarily close)**.** *If (X, d) is a metric space and the points $x, y \in X$ satisfy $\forall \varepsilon \in [0, \infty), [d(x, y) < \varepsilon]$, then $x = y$.*

Complete the proof here

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Theorem 2 (limits are unique)**.** *In a metric space* (X,d)*, if* (a_n) *is a convergent sequence and* $(a_n) \to a$ *and* $(a_n) \to a'$ *then* $a = a'$*.*

Hint 3. In order to prove this you should:

1. use the triangle inequality (M4) to usefully write $d(a,a') \leq$ something,
2. apply lemma 1.

Complete the proof here

$\blacksquare$

With our knowledge of convergence we can tie together our intuitions about eventually constant sequences and convergence.

Lemma 4 (eventually constant is convergent)**.** *If (X, d) is a metric space and (a_n) is an eventually constant sequence in (X, d), then (a_n) converges.*

Complete the proof here

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The converse of this is not generally true.

Question 5. Can you find a convergent sequence which is not eventually constant?

Complete the proof here

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There is, however, a case in which it is true.

Lemma 6 (convergent is constant in discrete)**.** *If (X, d) is a discrete metric space then every convergent sequence is eventually constant.*

Hint 7. What happens when $0 < \varepsilon < 1$ in a discrete metric space?

Complete the proof here

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Sometimes one comes across sequences which would converge if they could, but the point to which they would converge is not in the ambient metric space. Such a sequence does not satisfy the definition of convergence, but we do have a mathematical way of describing this situation.

Definition 8 (Cauchy sequence)**.** A sequence (a_n) in a metric space (X,d) is called a *Cauchy sequence* when it satisfies

$$\forall \varepsilon \in [0,\infty), \exists N \in \mathbb{N}, \forall n, m \in \mathbb{N}, [(n \geq N) \wedge (m \geq N) \rightarrow d(a_n, a_m) < \varepsilon] \quad .$$

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We think of these as "almost convergent" sequences, and, as we would hope, "almost convergent" is implied by actual convergence.

Lemma 9 (convergent is Cauchy)**.** *If (a_n) is a convergent sequence in a metric space (X,d) then (a_n) is Cauchy.*

Complete the proof here

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The following lemma explains the intuition that Cauchy sequences "would converge if they could, but sometimes the point to which they would converge is not in the ambient metric space."

Lemma 10. *Let (Y,d) be a sub-metric-space of a metric space (X,d). Suppose (a_n) is a sequence in (Y,d) that converges in (X,d) to a point $a \in X \setminus Y$, that is, a lies in X but not in the subspace Y. Then:*

1. *The sequence (a_n) is a Cauchy sequence in (Y,d).*
2. *The sequence (a_n) does not converge (Y,d).*

Hint 11. When proving this result, consider the following:

1. Recall that we defined a sequence in a subset of a metric space to be a sequence in the metric space whose every term is in the subset. Thus in particular, any sequence in a sub-metric-space is also a sequence in the ambient metric space.
2. For part 2, if (a_n) converged to a point b in the metric space Y, what could we say about the relationship between the points a and b?

Complete the proof here

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We have already suggested that not every Cauchy sequence is actually convergent. In a sense suggested by the examples below, Cauchy sequences which are not convergent detect 'holes' in our metric space.

Example 12 (*Cauchy is not necessarily convergent*)

For both of these examples, prove that the sequence is Cauchy but not convergent.

1. Let (a_n) be the sequence defined by $a_n :\equiv n/n+1$, in the set $(-\infty, 1) \cup (1, \infty) \subseteq \mathbb{R}$ with the Euclidean metric. The first few terms are $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

2. Let (b_n) be the sequence defined by $b_n :\equiv$ "the $(n+1)$-digit decimal expansion of $\sqrt{2}$", in the set $\mathbb{Q} \subseteq \mathbb{R}$ with the Euclidean metric. The first few terms are $1, 1.4, 1.41, 1.414, \ldots$ or $1, \frac{14}{100}, \frac{141}{100}, \frac{1414}{1000}, \ldots$

Complete the proof here

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Finally, it will prove useful later to understand how Cauchy sequences are sensitive to the finiteness of the space.

Lemma 13 (Cauchy sequence in finite space is eventually constant)**.** *Let (F, d) be a metric space, with F a finite set, and let (a_n) be a Cauchy sequence in F. Then (a_n) is eventually constant.*

Complete the proof here

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Corollary 14 (Convergent sequence in finite space is eventually constant)**.** *Every convergent sequence in a finite metric space is eventually constant.*

Complete the proof here

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