Metric Spaces Worksheet 2

Sequences I

We now turn our attention to a concept naturally suited to a phrasing within the language of metric spaces.

Definition 1 (sequence in a metric space)**.** A *sequence* in a metric space (X,d) is an ordinary function $a:\mathbb{N}\to X$. That is, it assigns to every natural number $n\in\mathbb{N}$ a point in $a(n)\in X$. By convention we write a_n for $a(n)$ and (a_n) for the sequence $a:\mathbb{N}\to X$. We may sometimes also speak of a sequence (a_n) *in a subset* $S\subseteq X$, by this we mean a sequence in X whose every term is in S, that is, $\forall n\in\mathbb{N}, a_n\in S$. ⌟

Example 2 (*some sequences in* $\mathbb{R}$)

Examples of sequences abound. In the Euclidean space $\mathbb{R}$, the following are all sequences:

1. (a_n) where $a_n :\equiv n$; the terms here are $0,1,2,3,\ldots$
2. (b_n) where $b_n :\equiv n/(1+n)$; the terms here are $0,\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots$
3. (c_n) where $c_n :\equiv (-1)^n$; the terms here are $1,-1,1,-1,\ldots$
4. (d_n) where $d_n :\equiv 2^n$ for $n\le 10$ and $d_n :\equiv 2048$ for $n>10$; the terms here are $1,2,4,8,\ldots,512,1024,2048,2048,\ldots$
5. (e_n) where $e_n :\equiv$ the 42^{nd} prime number; the terms here are $191,191,191,191,\ldots$

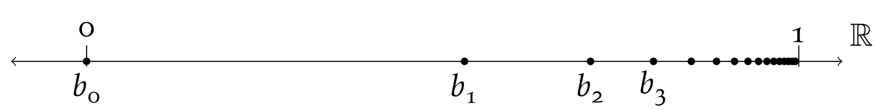

Figure 1: Depiction of the terms of the sequence of example 2 (2)

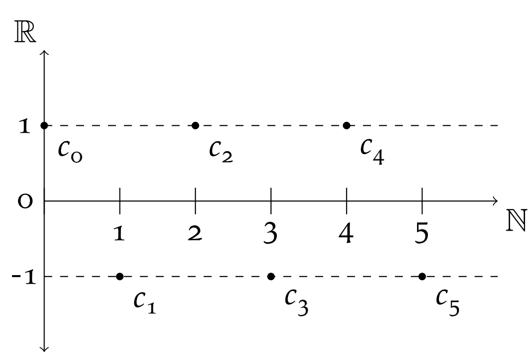

Figure 2: 2D depiction of the sequence of example 2 (3)

Some of the examples, in particular example 2 (4) and example 2 (5), have a special feature that we'll now isolate as an exercise in familiarising ourselves with sequences.

Definition 3 (constant and eventually constant sequences)**.** Given a sequence (a_n) in a metric space (X, d), we say that (a_n) is *constant* if $\forall n \in \mathbb{N}, a_n = a_0$. We call (a_n) *eventually constant* if

$$\exists N \in \mathbb{N}, \forall n \in \mathbb{N}, [n \geq N \to a_n = a_N] \quad .$$

⌟

So we can see that example 2 (5) is a constant sequence, and example 2 (4) is an eventually constant sequence with $N = 11$. These ideas are related to one another.

Lemma 4 (constant sequences are eventually constant)**.** *If (a_n) is a constant sequence in a metric space (X, d), then (a_n) is eventually constant.*

Complete the proof here

■

Example 2 (2) above is also of interest. By looking at the terms we see that the sequences approaches 1. In fact, we see that the sequence is arbitrarily close to 1 in the sense that it is eventually within any distance of 1 that we might care to choose. This leads us to our next sequence-related definition.

Definition 5 (convergent sequences)**.** A sequence (a_n) in a metric space (X,d) is said to *converge* to a point $a \in X$ when

$$\forall \varepsilon \in [0,\infty), \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, [n \geq N \to d(a_n, a) < \varepsilon] \quad .$$

In this case we write $\lim a_n = a$ or $(a_n) \to a$. If there is no point $a \in X$ such that $(a_n) \to a$ then we say that (a_n) is *divergent*. ⌟

We will shortly see that the sequence (b_n) of example 2 (2) is a convergent sequence, and so convergence does align with our intuitions at least in this case. Unfortunately the definition of divergence we've given is not so useful, and so to remedy that prove the following lemma.

Lemma 6 (divergent sequences)**.** *The statement "(a_n) is divergent in (X,d)" is logically equivalent to*

$$\forall a \in X, \exists \varepsilon \in [0,\infty), \forall N \in \mathbb{N}, \exists n \in \mathbb{N}, [(n \geq N) \wedge (d(a_n, a) \geq \varepsilon)] \quad .$$

This lemma helps us to read what divergence means: a sequence (a_n) is divergent when it always stays at least some distance from every point (where that distance may change depending on the point).

Complete the proof here

■

In these terms then, we may frame example 2 (2) and example 2 (1) as follows.

Proposition 7 ((n) is a divergent sequence)**.** *In the Euclidean metric space* $\mathbb{R}$*, the sequence* (a_n) *where* $a_n :\equiv n$ *is divergent.*

Complete the proof here

■

Proposition 8 ($(n/(1+n))$ converges to 1). *In the Euclidean metric space $\mathbb{R}$, the sequence (b_n) defined as $b_n :\equiv n/(1+n)$, converges to 1.*

Complete the proof here

$\blacksquare$